NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

COURSE MATERIALS

ECT 204:SIGNALS & SYSTEMS

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Electronics and Communication Engineering

  M.Tech in VLSI
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Providing Universal Communicative Electronics Engineers with corporate and social relevance towards sustainable developments through quality education.

DEPARTMENT MISSION

1) Imparting Quality education by providing excellent teaching, learning environment.

2) Transforming and adopting students in this knowledgeable era, where the electronic gadgets (things) are getting obsolete in short span.

3) To initiate multi-disciplinary activities to students at earliest and apply in their respective fields of interest later.

4) Promoting leading edge Research & Development through collaboration with academia & industry.

PROGRAMME EDUCATIONAL OBJECTIVES

PEOI. To prepare students to excel in postgraduate programmes or to succeed in industry/ technical profession through global, rigorous education and prepare the students to practice and innovate recent fields in the specified program/ industry environment.

PEO2. To provide students with a solid foundation in mathematical, Scientific and engineering fundamentals required to solve engineering problems and to have strong practical knowledge required to design and test the system.

PEO3. To train students with good scientific and engineering breadth so as to comprehend, analyze, design, and create novel products and solutions for the real life problems.

PEO4. To provide student with an academic environment aware of excellence, effective communication skills, leadership, multidisciplinary approach, written ethical codes and the life-long learning needed for a successful professional career.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: Facility to apply the concepts of Electronics, Communications, Signal processing, VLSI, Control systems etc., in the design and implementation of engineering systems.

PSO2: Facility to solve complex Electronics and communication Engineering problems, using latest hardware and software tools, either independently or in team.

COURSE OUTCOMES
ECT 204

SUBJECT CODE: ECT 204	
COURSE OUTCOMES After the completion of the course student will be able to:	
C204.1	Represent various signals and systems
C204.2	Represent & Analyze the continuous time system with Laplace transform and Fourier transform
C204.3	Understand the concept of sampling
C204.4	Analyze the discrete time system using DTFT
C204.5	Analyze the DT systems with Z Transform

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

CO'S	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C204.1	3	3										
C204.2	3	3	1	2								
C204.3	3	3		2								
C204.4	3	3	1	2								
C204.5	3	3	1	2								

CO'S	PSO1	PSO2
C204.1		1
C204.2	3	1
C204.3		1
C204.4	3	1
C204.5	3	1

SYLLABUS

Elementary signals, Continuous time and Discrete time signals and systems, Signal operations, Differential equation representation, Difference equation representation, Continuous time LTI Systems, Discrete time LTI Systems, Correlation between signals, Orthogonality of signals, Frequency domain representation, Continuous time Fourier series, Continuous time Fourier transform, Using Laplace transform to characterize Transfer function, Stability and Causility using ROC of Transfer transform, Frequency response, Sampling, Aliasing, Z transform, Inverse Z transform, Unilateral Z-transform, Frequency domain representation of discrete time signals, Discrete time Fourier series and discrete time Fourier transform (DTFT), Analysis of discrete time LTI systems using the above transforms.

Text Books

1. Alan V. Oppenheim and Alan Willsky, Signals and Systems, PHI, 2/e, 2009

2. Simon Haykin, Signals & Systems, John Wiley, 2/e, 2003

Reference Books

1. Anand Kumar, Signals and Systems, PHI, 3/e, 2013.

2. B P. Lathi, Priciples of Signal Processing & Linear systems, Oxford University Press.

3. Gurung, Signals and System, PHI.

4. Mahmood Nahvi, Signals and System, Mc Graw Hill (India), 2015.

5. P Ramakrishna Rao, Shankar Prakriya, Signals and System, MC Graw Hill Edn 2013.

6. Rodger E. Ziemer, Signals & Systems - Continuous and Discrete, Pearson, 4/e, 2013

Course Contents and Lecture Schedule

Module	Topic	Number of lecture hours
I	Elementary Signals, Classification and representation of continuous time and discrete time signals, Signal operations	4
	Continuous time and discrete time systems – Classification, Properties.	3
	Representation of systems: Differential equation representation of continuous time systems. Difference equation representation of discrete systems.	2
	Continuous time LTI systems and convolution integral.	2
	Discrete time LTI systems and linear convolution.	2
	Stability and causality of LTI systems.	2
	Correlation between signals, Orthogonality of signals.	1
II	Frequency domain representation of continuous time signals - continuous time Fourier series and its properties.	4
	Continuous time Fourier transform and its properties. Convergence and Gibbs phenomenon	3
	Review of Laplace Transform, ROC of Transfer function, Properties of ROC, Stability and causality conditions.	3
	Relation between Fourier and Laplace transforms.	1
III	Analysis of LTI systems using Laplace and Fourier transforms. Concept of transfer function, Frequency response, Magnitude and phase response.	4
	Sampling of continuous time signals, Sampling theorem for lowpass signals, aliasing.	3
IV	Frequency domain representation of discrete time signals, Discrete time fourier series for discrete periodic signals. Properties of DTFS.	4
	Discrete time fourier transform (DTFT) and its properties. Analysis of discrete time LTI systems using DTFT. Magnitude and phase response.	5
V	Z transform, ROC , Inverse transform, properties, Unilateral Z transform.	3
	Relation between DTFT and Z-Transform, Analysis of discrete time LTI systems using Z transforms, Transfer function. Stability and causality using Z transform.	4

QUESTION BANK

MODULE 1

1. Check whether the signal $x(t)=10\sin 50\Pi t + \cos 100\Pi t$ is periodic or not. Find the fundamental period if periodic.
2. Plot the signal $x(t)=2u(t+1)+2u(t)-3u(t-3)-2u(t-5)$
3. Check whether the following signals are energy or power signals

   (a) $x(t)=e^{-3|t|}$

   (b) $x(n)=(1/4)^n u(n)$

   (c) $x(t)=e^{3t}u(t-2)$

   (d) $x(n)=(1/2)^n u(n-2)$

   (e) $x(t)=e^{-2t}u(-t)$

4. Determine whether the system, $y(n)=x(n)+5/x(n-5)$ is Time invariant, Linear,Static,Stable and Causal.

5. Show that the product of 2 odd signals is an even signal

6. What are signals? Explain the classification of signals.

7. Explain about the different properties of system.

8. Define static and dynamic system.

9. Define odd and even signal.

10. Find the odd and even components of the signal $x(n)=\{1,2,-1\}$

11. Check the causality and stability of the systems whose impulse responses are given by (i) $h(t)= e^{at} u(t)$ (ii) $h(n)= 2^n u(-n)$

12. Find the convolution of $x(t)=u(t+1)-u(t-1)$
with $h(t)=u(t+2)-u(t-2)$

13. Find the output of the system with impulse response $h(n)=\{1,2\}$ to the input $x(n)= \{2,-3,7\}$

14. Compute the auto correlation of the signal $x(n)=a^n u(n)$ for $0<a<1$

15. Find the output of the system with impulse response $h(n)= (1/2)^n u(n)$ to the input $x(n)= u(n-5)$

16. Find the convolution of $x_1(t)=e^{-2t}u(t)$ and $x_2(t)= u(t) + u(t-2)$

17. Find the convolution of $h(n)=\{1,2\}$ with $x\{n\}=\{2,-3,7\}$ using Matrix method.

MODULE II

1. Determine the Fourier transform and Laplace transform of $x(t)=\delta(t)$
2. Find the Laplace transform of $tu(t)$
3. Find the inverse Laplace of $X(s)=(2/s^2)-4$
4. Find the Laplace transform of (a) $u(t)$ (b) Impulse function
5. Find the inverse Laplace transform of $1/[s(s-3)]$

4. Determine the transfer function of system with poles at s=-1,2 and zeros at s=3

5. Find the inverse Laplace transform of $1/[s^2-4s+3]$

6. Prove Parsevals Theorem

7. Determine the unilateral laplace transform of sinwt and coswt

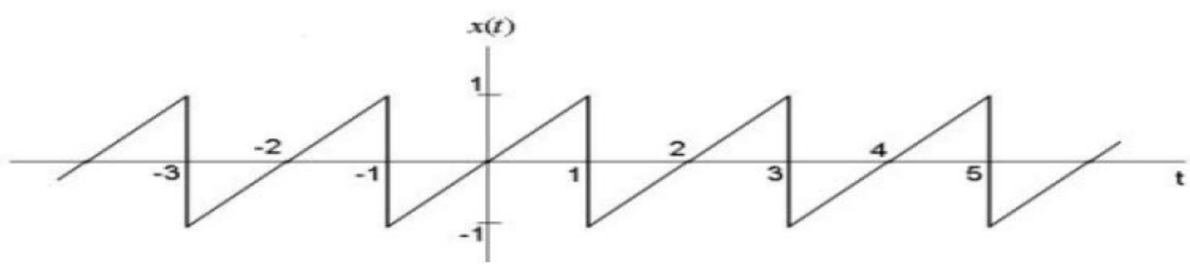
8. State and Prove the properties of Laplace transform.

9. What is ROC

10. Find the Laplace transform and ROC of the signal $x(t)= -e^{at}u(-t)$

11. Find the Fourier transform of the signal $x(t)=e^{-a|t|}$

12. Obtain the Trigonometric Fourier Series representation of the signal



13. What is the relation between Laplace and Fourier transform.

14. State and Prove the properties of CTFS.

15. State and Prove the properties of CTFT.

MODULE III

1. State and prove the sampling theorem for low pass signals

2. A signal x(t) = 2 cos 400πt + 6 cos 600 πt is sampled with a sampling frequency 800Hz. Write the resultant discrete time signal.

3. Determine the Nyquist rate of sampling for the signals

   i) x(t) = 2sin 250πt + 3cos ^{2}500t

   ii) x(t) = 10 sinc 500t

4. The step response of an LTI system is **$(1-e^{-t}-te^{-t})u(t)$**. For an input x(t), the output is observed to be **$(2-3e^{-t}+e^{-3t})u(t)$** . For this observed measurement, determine the input to the system using laplace transform.

5. For the following system described by differential equation, find the impulse response if the system is stable

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{d^2x(t)}{dt^2}+8\frac{dx(t)}{dt}+13x(t)$$

   Assume initial conditions as zero.

6. A continuous time LTI system is described by the differential equation

$$\frac{d}{dt}y(t)+5y(t)=x(t)$$

Determine the response of the system to the input $x(t)=e^{-2t}u(t)$ using Fourier Transform.

7. Explain the Dirichlet's condition for the existence of Fourier Transform

MODULE IV

1. Evaluate the inverse Z-transform of

$$X(z) = \log \frac{1}{1-az^{-1}} \qquad |a|<|z|$$

2. Evaluate the DTFT of following signal

$$x(n) = a^n \sin \Omega_0 n u(n)$$

3. Find the DTFT of $x(n) = 0.25^n u(n+2)$

4. Give the Parseval's theorem for DTFT. Prove it.

5. Compute the energy of the sequence

$$x(n) = \frac{\sin \quad \Omega_c n}{\pi n}$$

6. An LTI system is characterized by the system function given as

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

   Under what conditions the system will be obey causality and stability?

   Determine the impulse response of the system such that

   i) The system is causal ii) The system is stable

   Justify the answers.

7. Find the z-transform and specify ROC

   i) $x(n) = u(n-2) * (\frac{2}{3})^n u(n)$

   ii) $x(n) = -n(\frac{1}{3})^n u(-n-1)$

8. Write the Fourier series representation of a discrete time periodic signal with periodicity N. What is the difference between continuous time and discrete time

   Fourier series?

MODULE V

1. A system is described by the difference equation

$$y(n) = x(n) - x(n-1) - \frac{1}{4}y(n-1) + \frac{1}{8}y(n-2)$$

   Determine the impulse response of the system using fourier transform. Also find the step response of the system.

2. The frequency response of a three point moving average system is given as

$$H(e^{j\Omega}) = \frac{1}{4}\ (1 + \cos\ \ \Omega)\ e^{-j\Omega}.$$

   Determine the difference equation representation of the system.

3. Determine the response of the system with impulse response h(n)= 0.5^n u(n) to the input

$$x(n) = 10 - 5\sin\ \ \frac{\pi}{2}n$$

4. An LTI system is described by the difference equation

$$y(n) - \frac{9}{4}y(n-1) + \frac{1}{2}y(n-2) = x(n) - 3x(n-1)$$

   Specify the ROC of H(z), and determine h(n) for the following conditions

   i) The system is stable ii) The system is causal

5. A system is described by the difference equation

$$y(n) = x(n) - x(n-1) - \frac{1}{4}y(n-1) + \frac{1}{8}y(n-2)$$

   Determine the transfer function of the system using Z transform.

6. A system is described by the difference equation

$$y(n) = x(n) - x(n-1) - \frac{1}{4}y(n-1) + \frac{1}{8}y(n-2)$$

   Determine the frequency response of the system using fourier transform

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NOTES

MODULE 1

Signals:

* It is a physical quantity that varies with time, space or any other independent variables.
* Signal represents data. ie data is encoded by means of a signal.
* It is a function of time: horizontal axis represents time and vertical axis represents amplitude.

Examples:

Speech signal (one dimensional) that describes the aquastic pressure variation as a function of time, t.

picture signal (Two dimensional) that describes the gray level as a function of spatial co-ordinates x & y.

* If a signal depends on only one variable, then it is known as one dimensional signal and a signal depends on two variables, then it is known as two dimensional signal.

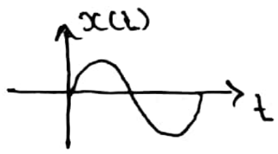
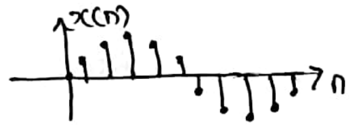
Classification of signals:

Signals are mainly classified into two:

* Continuous time signals
* Discrete time signals.

They are further classified as.

* Deterministic / non deterministic.
* periodic / aperiodic
* Even / odd. * Energy / power

Continuous $x(t)$	discrete. $x(n)$
* They are defined for every value of time t	They are defined at specific interval of time.
* Take all possible values of amplitude	Take finite set of amplitude values.
	

NOTE: A discrete time signal is obtained by sampling a continuous time signal at regular intervals.

$$x(nT) = x(n) = x(t)\Big|_{t=nT}$$

where T is the sampling period and n is the integer ranging from $-\infty$ to $+\infty$ called time index.

Sampling is the process of converting a continuous signal into discrete time signal.

1. Sketch the continuous time signal $x(t) = 2\sin\pi t$ for interval $0 \le t \le 2$. Sample the signal with a sampling period $T = 0.2\,s$ and then sketch the discrete time signal.

Given $x(t) = 2\sin\pi t$

$x(0) = 0$

$x(0.25) = 1.414$

$x(0.5) = 2$

$x(0.75) = 1.414$

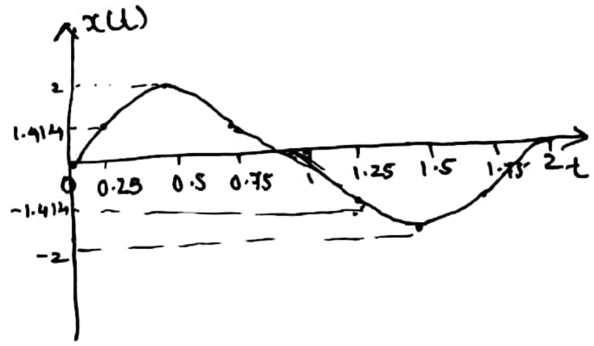
$x(1) = 0$

$x(1.25) = -1.414$

$x(1.5) = -2$

$x(1.75) = -1.414$

$x(2) = 0.$



$$x(nT) = x(n) = x(t)\big|_{t=nT}$$

$$= 2\sin\pi t\big|_{t=nT,\ t=0.2n}$$

$$= 2\sin(n\pi t)$$

$$= 2\sin(0.2\pi n)$$

$x(0) = 0$

$x(1) = 1.175$

$x(2) = 1.902$

$x(3) = 1.902$

$x(4) = 1.175$

$x(5) = 0$

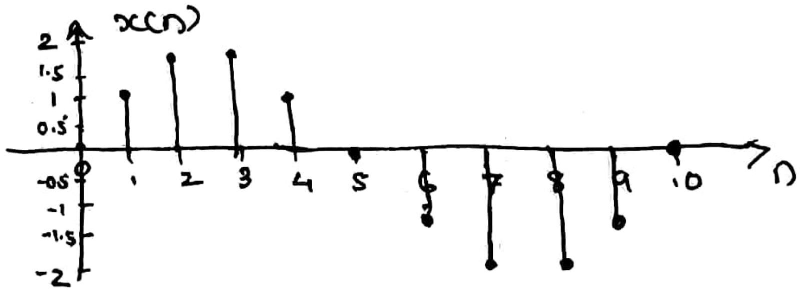
$x(6) = -1.175$

$x(7) = -1.902.$

$x(8) = -1.902.$

$x(9) = -1.175$

$x(10) = 0.$



2 Sketch the signal $x(t) = e^{-t}$ for an interval $0 \le t \le 2$. Sample the signal with a sampling period $T = 0.2\,s$ and then sketch the discrete time signal.

Given $x(t) = e^{-t}$

$x(0) = 1$

$x(0.25) = 0.77$

$x(0.5) = 0.606$

$x(0.75) = 0.472$

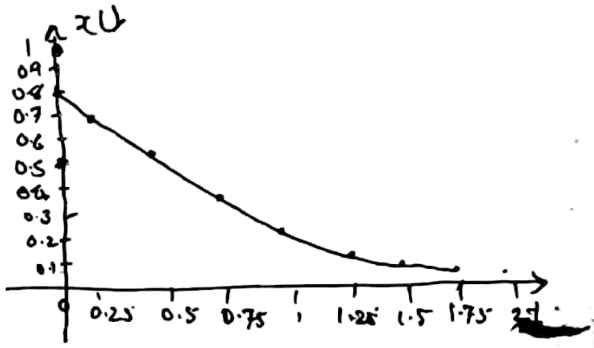
$x(1) = 0.367$

$x(1.25) = 0.286$

$x(1.5) = 0.223$

$x(1.75) = 0.173$

$x(2) = 0.135$



$$x(nT) = x(n) = x(t)\Big|_{t=nT}$$

$$= e^{-t}\Big|_{t = n \times 0.2}$$

$$= e^{-0.2n}.$$

$x(0) = 1$

$x(1) = e^{-0.2} = 0.818$

$x(2) = e^{-0.4} = 0.670$

$x(3) = e^{-0.6} = 0.54$

$x(4) = e^{-0.8} = 0.44$

$x(5) = e^{-1} = 0.36$

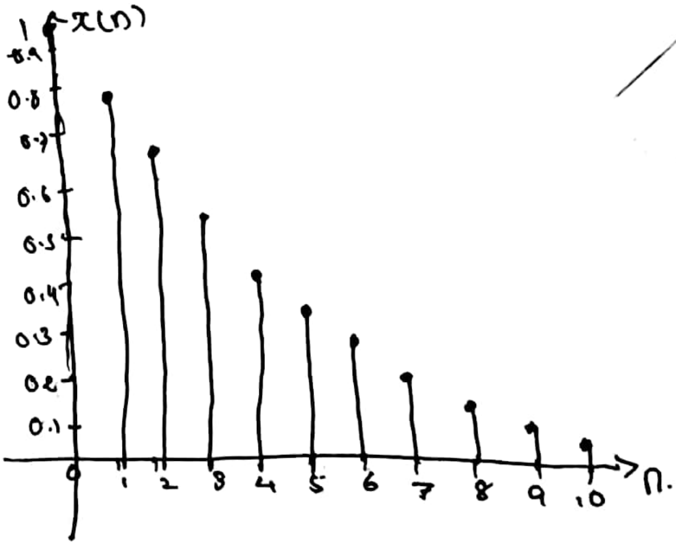
$x(6) = e^{-1.2} = 0.301$

$x(7) = e^{-1.4} = 0.246$

$x(8) = e^{-1.6} = 0.2$

$x(9) = e^{-1.8} = 0.165$

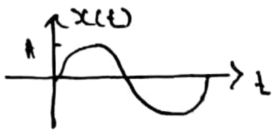
$x(10) = e^{-2} = 0.135.$



Deterministic / non deterministic signal:

Deterministic signal is known for all time and can be predicted in advance exactly. ie everything is known about the signal.

Eg: Sine wave with known phase.



non deterministic:

Some parameter of the signal is unknown and cannot be predicted exactly.

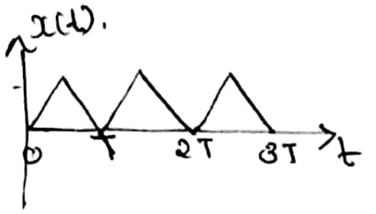
Eg: Noise signal : we can't define the amplitude values of a noise signal by means of formula or function.

Since it is impossible to specify their behaviour in terms of function, such signals are described by expected values such as mean and variance.

periodic / aperiodic signal:

A continuous time signal $x(t)$ is said to be periodic with period T, if there is a positive value of T for which $x(t+T) = x(t)$ for all t $\rightarrow$ ①.

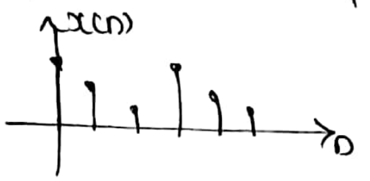
The smallest positive value of T for which eqn ① hold is known as fundamental period.



A signal is aperiodic or non periodic if the condition in eqn ① is not satisfied for atleast one value of t.

A discrete time signal x(n) is said to be periodic with period N, if there is a positive value of N for which $x(n+N) = x(n)$ for all $n \rightarrow$ ②.

The smallest positive value of N for which eqn ② hold is known as fundamental period.



Here the sequence is repeating after every 3 samples.

∴ fundamental period = 3.

If eqn ② does not satisfy for atleast one value of n, then the discrete time signal is aperiodic.

* The sum of two periodic signals $x_1(t)$ & $x_2(t)$ with period T_1 & T_2 is said to be periodic iff. the ratio T_1/T_2 is a rational number, otherwise the sum is non periodic.

Qn. Determine whether the following signals is periodic. If a signal is periodic determine its fundamental period

a) $x(t) = \cos(t + \pi/4)$

ω - coefficient of t

$T = \frac{2\pi}{\omega} = 2\pi$ secs.

∴ the given signal is periodic.

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b) $x(t) = \underbrace{\cos(\pi/3\, t)}_{x_1(t)} + \underbrace{\sin(\pi/4\, t)}_{x_2(t)}$

$\omega_1 = \pi/3$ $\qquad$ $\omega_2 = \pi/4$

$T_1 = \frac{2\pi}{\omega_1}$ $\qquad$ $T_2 = \frac{2\pi}{\omega_2}$

$= 2\pi/\pi/3 = 6.$ $\qquad$ $= 2\pi/\pi/4 = 8$

$\frac{T_1}{T_2} = \frac{6}{8} = \frac{3}{4}$

$\frac{T_1}{T_2} = \frac{3}{4}$ is a rational number

$\therefore$ the given signal is periodic.

Fundamental period, $T = 4T_1$ or $3T_2$

$= 4 \times 6$ $\qquad$ 3×8

$= 24$ secs.

c) $x(t) = \underbrace{\cos t}_{x_1(t)} + \underbrace{\sin \sqrt{2}\, t}_{x_2(t)}$

$\omega_1 = 1$ $\qquad$ $\omega_2 = \sqrt{2}$

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{1}$ $\qquad$ $T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

$\frac{T_1}{T_2} = \frac{2\pi}{\sqrt{2}\pi} = \sqrt{2}$

$\frac{T_1}{T_2} = \sqrt{2}$ is irrational $\therefore$ the given signal is aperiodic.

d) $x(t) = \sin^2 t$

$= \frac{1 - \cos 2t}{2}.$

$\omega_1 = 2$

$T = \frac{2\pi}{\omega} = \pi$ is rational number $\therefore$ the given signal is periodic with period π secs.

H.W a) $x(t) = e^{j(\pi/2 t - 1)}$

$\omega = \pi/2$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$

$\therefore$ the given signal is periodic with period 4 secs.

b) $x(t) = 2\cos(10t+1) - \sin(4t-1)$

$\omega_1 = 10$ $\omega_2 = 4$

$T_1 = \pi/5$ $T_2 = \pi/2$

$\frac{T_1}{T_2} = \frac{2}{5}$ is a rational no. $\therefore$ periodic

$T = 5T_1 = \pi$ secs.

c) $x(t) = \cos 60\pi t + \sin 50\pi t$

$T_1 = 1/30$ $T_2 = \frac{1}{25}$

$\frac{T_1}{T_2} = \frac{5}{6}$ rational no. $\therefore$ periodic

$T = 6T_1 = 6/30 = 1/5$ secs.

d) $x(t) = 3\cos 4t + 2\sin \pi t$

$T_1 = \pi/2$ $T_2 = 2$.

$\frac{T_1}{T_2} = \pi/4$ rational no. $\therefore$ periodic

$T = 4T_1 = 4\,\pi/2 = 2\pi$ secs.

e) $x(t) = \cos(1/3\, t) + \sin(1/4\, t)$.

$T_1 = 6\pi$ $T_2 = 8\pi$

$\frac{T_1}{T_2} = 3/4$ rational no. $\therefore$ periodic

$T = 4T_1$

$= 4 \times 6\pi = 24\pi$ secs.

Qn. Find whether the following discrete time signal is periodic or not.

a) $x(n) = e^{j6\pi n}$

$\Omega = 6\pi$

NOTE: If Ω is a multiple of π, then the signal is periodic otherwise aperiodic.

$\Omega = 6\pi$ is a multiple of π $\therefore$ the given signal is periodic

$$N = \frac{2\pi}{\Omega} \cdot m$$

$$= \frac{2\pi}{6\pi} \cdot m$$

$$= \frac{1}{3} \cdot m$$

$$= \frac{1}{3} \ (m = 3)$$

$$= \cancel{\frac{3}{3}}\ 1$$

b) $x(n) = e^{j3/5(n + 1/2)}$

$\Omega = 3/5$ is not a multiple of π.

$\therefore$ the given signal is aperiodic

c) $x(n) = \cos \frac{2\pi}{3} n$

$\Omega = \frac{2\pi}{3}$ is a multiple of π $\therefore$ periodic.

$$N = \frac{2\pi m}{\Omega} = \frac{2\pi}{2\pi/3} m$$

$$= 3m$$

$$= 3 \ (m = 1).$$

d) $x(n) = \cos \pi/3\, n + \cos 3\pi/4\, n$

$\Omega_1 = \pi/3 \qquad \Omega_2 = 3\pi/4$

Ω_1 & Ω_2 are multiple of π $\therefore$ periodic.

$N_1 = \frac{2\pi}{\Omega_1} \cdot m \qquad N_2 = \frac{2\pi}{\Omega_2} m$

$= \frac{2\pi}{\pi/3} m \qquad = \frac{2\pi}{3\pi/4} m$

$= 6m \qquad = \frac{8}{3} m$

$= 6 \ (m=1) \qquad = 8 \ (m=3)$

$\frac{N_1}{N_2} = \frac{6}{8} = \frac{3}{4}$

$N = 4N_1 = 4 \times 6 = 24$ secs.

e) $x(n) = 5 \cos(0.2\pi n)$

$\Omega = 0.2\pi$

is a multiple of π

$\therefore$ periodic.

$N = \frac{2\pi}{\Omega} \cdot m$

$= \frac{2\pi}{0.2\pi} m = 10\, m$

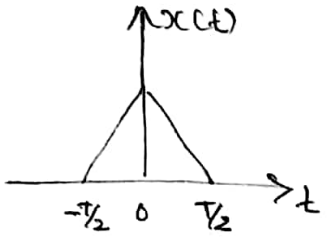
$= 10 \ (m=1)$

Even / odd signals:

Symmetric   Anti symmetric.

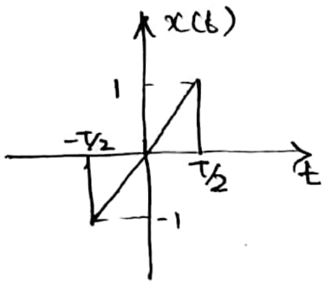
A signal $x(t)$ or $x(n)$ is referred to as even signal

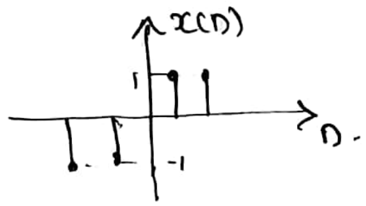
if $\quad x(-t) = x(t)$
$\quad\quad x(-n) = x(n)$.



A signal $x(t)$ or $x(n)$ is referred to as odd signal

if $\quad x(-t) = -x(t)$
$\quad\quad x(-n) = -x(n)$.





Any signal $x(t)$ can be ~~rep.~~ expressed as sum of even and odd components. ie $x(t) = x_e(t) + x_o(t) \longrightarrow ①$

when $x_e(t)$ - even part of $x(t)$
$\quad\quad x_o(t)$ - odd part of $x(t)$

$$x(-t) = x_e(-t) + x_o(-t)$$

$$x(-t) = x_e(t) - x_o(t) \longrightarrow ②$$

$① + ② \Rightarrow \quad x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = 2x_e(t)$

$$\therefore x_e(t) = \frac{x(t) + x(-t)}{2}.$$

$① - ② \Rightarrow \quad x_o(t) = \dfrac{x(t) - x(-t)}{2}.$

||ly for discrete time s/l

$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

Qn. Find the even and odd components of a) $x(t) = e^{jt}$

$$x_e(t) = \frac{x(t)+x(-t)}{2} = \frac{e^{jt}+e^{-jt}}{2} = \cos t$$

$$x_o(t) = \frac{x(t)-x(-t)}{2} = \frac{e^{jt}-e^{-jt}}{2} = j\sin t$$

b) $x(t) = \cos t + \sin t + \cos t \sin t$

$x(-t) = \cos(-t) + \sin(-t) + \cos(-t)\sin(-t)$

$x(-t) = \cos t - \sin t - \cos t \sin t$

$$x_e(t) = \frac{x(t)+x(-t)}{2} = \frac{\cos t + \cancel{\sin t} + \cancel{\cos t \sin t} + \cos t - \cancel{\sin t} - \cancel{\cos t \sin t}}{2}$$

$$= \cos t$$

$$x_o(t) = \frac{x(t)-x(-t)}{2} = \frac{\cancel{\cos t} + \sin t + \cos t \sin t - \cancel{\cos t} + \sin t + \cos t \sin t}{2}$$

$$= \frac{2\sin t + 2\cos t \sin t}{2} = \sin t + \cos t \sin t$$

$$= \sin t\,[1+\cos t]$$

Qn. Find the even and odd components of $x(n) = \{\underset{x(-2)}{-2}, \underset{x(-1)}{1}, \underset{\uparrow x(0)}{2}, \underset{x(1)}{-1}, \underset{x(2)}{3}\}$

$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$x_e(0) = \frac{1}{2}[x(0) + x(-0)] = \frac{1}{2}[2+2] = 2.$

$x_e(1) = \frac{1}{2}[x(1) + x(-1)] = \frac{1}{2}[-1+1] = 0.$

$x_e(2) = \frac{1}{2}[x(2) + x(-2)] = \frac{1}{2}[3+-2] = 1/2$

$x_e(-1) = x_e(1) = 0 \qquad x_e(-2) = x_e(2) = 1/2.$

$\therefore x_e(n) = \{1/2, 0, \underset{\uparrow}{2}, 0, 1/2\}$

$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

$x_o(0) = \frac{1}{2}[x(0) - x(-0)] = \frac{1}{2}[2-2] = 0$

$x_o(1) = \frac{1}{2}[x(1) - x(-1)] = \frac{1}{2}[-1-1] = -1$

$x_o(2) = \frac{1}{2}[x(2) - x(-2)] = \frac{1}{2}[3--2] = 5/2.$

$x_o(-1) = -x_o(1) = 1 \qquad x_o(-2) = -x_o(2) = -5/2.$

$\therefore x_o(n) = \{-5/2, +1, 0, -1, 5/2\}$

Qn. Show that the product of two even signals is an even signal.

Consider two signals $x_1(t)$ & $x_2(t)$

Let $x(t) = x_1(t) \cdot x_2(t)$.

$$\left.\begin{matrix} x_1(t) \\ x_2(t) \end{matrix}\right\} \text{even.}$$

$\therefore x(-t) = x_1(-t) \cdot x_2(-t)$

$= x_1(t) \cdot x_2(t) = x(t)$

$x(-t) = x(t)$

$\therefore$ the product of two even signals is an even signal.

Qn. Show that the product of two odd signals is an even signal.

Consider two signals $x_1(t)$ & $x_2(t)$.

$x(t) = x_1(t) \cdot x_2(t)$.

$x_1(t)$ & $x_2(t) \longrightarrow$ odd.

$x(-t) = x_1(-t) \cdot x_2(-t)$

$= -x_1(t) \cdot -x_2(t) = x_1(t) \cdot x_2(t) = x(t)$.

$x(-t) = x(t)$.

$\therefore$ The product of two odd signals is an odd signal.

Qn. S.T the product of an even and odd signal is an odd signal.

$x(t) = x_1(t) \cdot x_2(t)$.

$x_1(t)$ — even.

$x_2(t)$ — odd

$\therefore x(-t) = x_1(-t) \cdot x_2(-t)$

$= x_1(t) \cdot - x_2(t) = -x_1(t) \cdot x_2(t) = -x(t)$

$x(-t) = -x(t)$

$\therefore$ the product of an even and odd signal is an odd signal.

Energy / power signals :

For a signal $x(t)$, the total energy is defined as

$$E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \quad \text{Joules}$$

and average power, $P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$ Watts.

For a signal $x(n)$, the total energy is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2.$$

and average power, $P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2.$

* A signal $x(t)$ or $x(n)$ is called an energy signal if the energy satisfies the condition $0 < E < \infty$. For an energy signal power $= 0$.

* A signal $x(t)$ or $x(n)$ is called a power signal if the power P satisfies the condition $0 < P < \infty$. For a power signal $E = \infty$.

Energy signal $\begin{cases} E - \text{finite} \\ P = 0 \end{cases}$

power signal $\begin{cases} P - \text{finite.} \\ E = \infty. \end{cases}$

Qn. check whether the signal $x(t) = e^{-3t} u(t)$ is energy or power signal.

$$x(t) = e^{-3t} u(t)$$

$$E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \underset{T\to\infty}{Lt} \int_{-T}^{T} |e^{-3t} u(t)|^2 dt$$

$$= \underset{T\to\infty}{Lt} \int_{-T}^{T} e^{-6t} u(t) dt$$

$$= \underset{T\to\infty}{Lt} \int_{0}^{T} e^{-6t} dt$$

$$= \underset{T\to\infty}{Lt} \left[\frac{e^{-6t}}{-6} \right]_0^T$$

$$= \frac{1}{-6} \underset{T\to\infty}{Lt} \left[e^{-6T} - e^{-0} \right]$$

$$= \frac{-1}{6} \underset{T\to\infty}{Lt} \left[e^{-6T} - 1 \right] = \frac{-1}{6}\left[e^{-\infty} - 1 \right] = \frac{-1}{6}\left[0 - 1 \right] = \frac{1}{6} J$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |e^{-3t} u(t)|^2 dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{0}^{T} e^{-6t} dt = \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\frac{e^{-6t}}{-6} \right]_0^T$$

$$= \frac{-1}{6} \underset{T\to\infty}{Lt} \frac{1}{2T} \left[e^{-6T} - e^{-0} \right]$$

$$= \frac{-1}{6} \underset{T\to\infty}{Lt} \frac{1}{2T} \left[e^{-6T} - 1 \right] = \frac{-1}{6} \frac{1}{2\infty} \left[e^{-\infty} - 1 \right]$$

$$= \frac{1}{6} \frac{1}{\infty} = \frac{1}{\infty} = 0$$

∴ Energy s/l.

Qn. check whether the following signal is energy or power.

$x(n) = (1/3)^n u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2.$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

$$= \sum_{n=-\infty}^{\infty} |(1/3)^n u(n)|^2$$

$$= \sum_{n=0}^{\infty} |(1/3)^n|^2 = \sum_{n=0}^{\infty} (1/9)^n$$

$$= \frac{1}{1-1/9} = \frac{1}{\frac{9-1}{9}} = 1/8/9 = 9/8 \text{ J}.$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2.$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |(1/3)^n u(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (1/9)^2.$$

$1 - \frac{1}{9}$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1-(1/9)^{N+1}}{1-1/9}$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{9}{8} (1-(1/9)^{N+1})$$

$= 0$ //.

$\therefore$ Energy signal.

Qn. Find the power of the signal $x(n) = u(n)$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2,$$

$$\sum_{n=0}^{2} 1 = 1+1+1 = 3$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |u(n)|^2$$

$$\sum_{n=0}^{N} 1 = N+1$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} 1.$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \; N+1$$

$$= \underset{N\to\infty}{Lt} \frac{\cancel{N}(1+\frac{1}{N})}{\cancel{N}(2+1/N)}.$$

$$= \frac{1+\frac{1}{\infty}}{2+\frac{1}{\infty}} = \frac{1+0}{2+0} = \frac{1}{2} \text{ watts}$$

Qn Find the energy of the signal $x(n) = u(n)$.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2,$$

$$= \sum_{n=0}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} |u(n)|^2 = \sum_{n=0}^{\infty} 1^2$$

$$= \sum_{n=0}^{\infty} 1$$

$$= \infty.$$

Qn What is the total energy of the signal $x(n)$ which takes the values of unity at $n = -1, 0, 1$.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-1}^{1} 1^2 = \sum_{n=-1}^{1} 1 = 1+1+1$$

$$= \underline{\underline{3\ J}}$$

Qn. check whether the following signal is energy or power.

$x(t) = e^{j(2t+\pi/4)}$.

$$E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \underset{T\to\infty}{Lt} \int_{-T}^{T} \left| e^{j(2t+\frac{\pi}{4})} \right|^2 dt$$

$e^{j2t} = \cos 2t + j\sin 2t$

$|e^{j2t}| = \sqrt{\cos^2 2t + \sin^2 2t}$

$= \sqrt{1} = 1$

$$= \underset{T\to\infty}{Lt} \int_{-T}^{T} |e^{j2t}|^2 |e^{j\frac{\pi}{4}}|^2 \, dt$$

$$= \underset{T\to\infty}{Lt} \int_{-T}^{T} 1^2 . 1^2 \, dt = \underset{T\to\infty}{Lt} \int_{-T}^{T} dt = \underset{T\to\infty}{Lt} [t]_{-T}^{T}$$

$$= \underset{T\to\infty}{Lt} [T - -T].$$

$$= \underset{T\to\infty}{Lt} 2T = \infty //$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \left| e^{j(2t+\frac{\pi}{4})} \right|^2 dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |e^{j2t}|^2 \, |e^{j\pi/4}|^2 \, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} 1^2 . 1^2 \, dt = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} [t]_{-T}^{T} = \underset{T\to\infty}{Lt} \frac{1}{\cancel{2T}} . \cancel{2T} = 1.$$

$\therefore$ power signal.

Qn. A pair of sinusoidal signals with a common angular frequency is defined by $x_1(n) = \sin 5\pi n$ and $x_2(n) = \sqrt{3}\cos 5\pi n$.

a) Specify the condition which the period N of both $x_1(n)$ and $x_2(n)$ must satisfy for them to be periodic. 3

b) Evaluate the amplitude of the composite sinusoidal signal $y(n) = x_1(n) + x_2(n)$.

a) $\Omega_1 = 5\pi \quad \Omega_2 = 5\pi$

$N = \frac{2\pi \cdot m}{\Omega} \Rightarrow N = \frac{2\pi}{5\pi} m = 2 \ (m = 5).$

For $x_1(n)$ & $x_2(n)$ to be periodic their period N must be an integer. This can only be satisfied for $m = 5$.

b) $y(n) = x_1(n) + x_2(n)$

$= \sin 5\pi n + \sqrt{3}\cos 5\pi n.$

$= 2\left(\frac{1}{2}\sin 5\pi n + \frac{\sqrt{3}}{2}\cos 5\pi n\right).$

$= 2(\sin 30 \sin 5\pi n + \cos 30 \cos 5\pi n).$

$=$

Amplitude A is given by.

$A = \sqrt{(\text{amplitude of } x_1(n))^2 + (\text{amp of } x_2(n))^2}$

$= \sqrt{1 + 3} = 2$

Basic operations on signals:

Operation performed on the independent variable.

→ Time ~~scaling~~ shifting

→ Time scaling

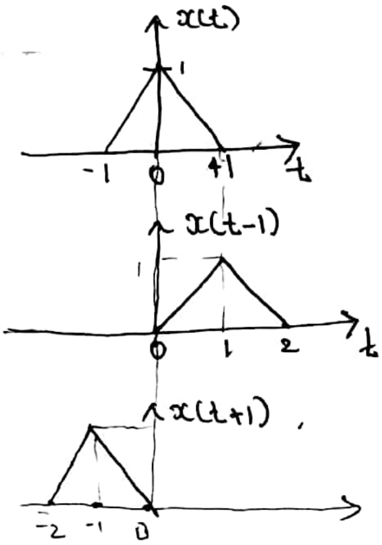
→ Reflection or Time folding.

1) Time shifting: Time shifting of $x(t)$ may delay or advance the signal in time.

Let $x(t)$ be a continuous time signal, replacing t by $(t+b)$ results in a time shifted signal $y(t)$ defined as, $y(t) = x(t+b)$.

If $b<0$ (-ve) $x(t)$ is shifted to right (delay) by an amount 'b' secs.

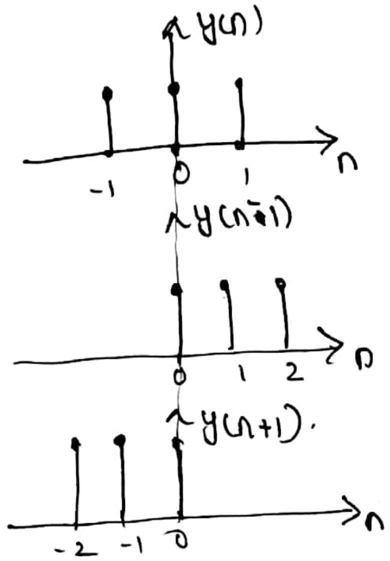
If $b>0$ (+ve), $x(t)$ is shifted to left (advance) by an amount 'b' secs.



Similarly, the time shifting operation of a discrete time signal is represented by $y(n) = x(n+k)$.

If $k<0$, $x(n)$ is shifted to right by amt. 'K' secs.

If $k>0$, $x(n)$ is shifted to left "

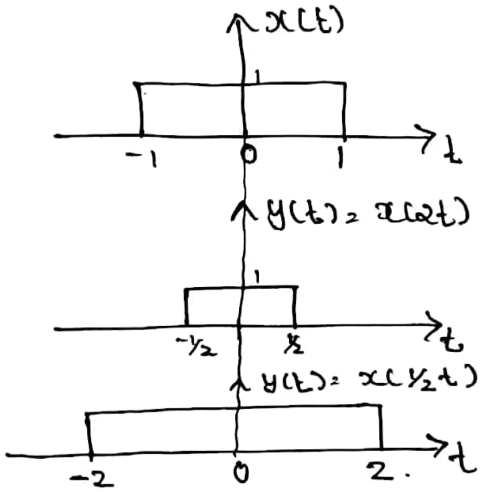


Time scaling:

Time scaling of signal $x(t)$ is $y(t) = x(at)$

If $a > 1$, then $y(t)$ is the compressed version of $x(t)$.

If $a < 1$, then $y(t)$ is the expanded version of $x(t)$.

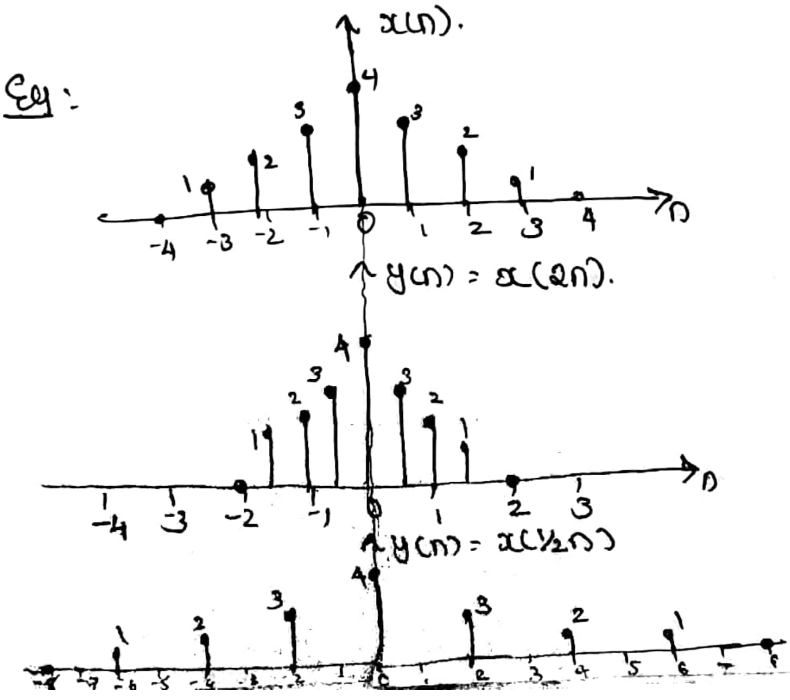


lly Time scaling of dis. time s/l $x(n)$ is
$y(n) = x(an)$.

If $a > 1$, then $y(n)$ is the compressed version of $x(n)$

If $a < 1$, then $y(n)$ is the expanded version of $x(n)$.

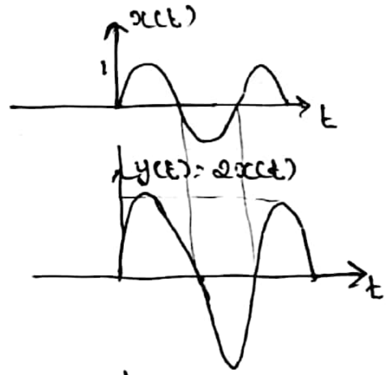
Eg:

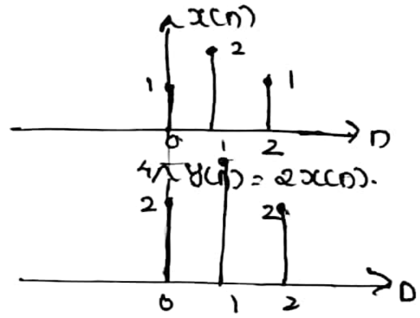


Operations performed on dependent variables.

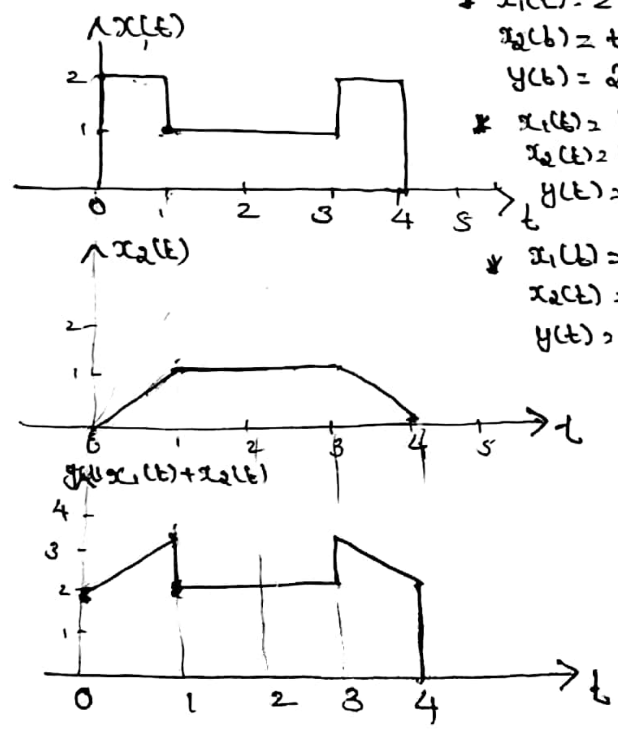
→ Amplitude scaling:
→ addition
→ multiplication

Amplitude scaling: Amplitude scaling of a continuous time signal $x(t)$ can be represented by $y(t) = A\,x(t)$.
and that of a discrete time signal $x(n)$ can be represented by $y(n) = A\,x(n)$. If $A > 1 \rightarrow$ amplification
If $A < 1 \longrightarrow$ attenuation

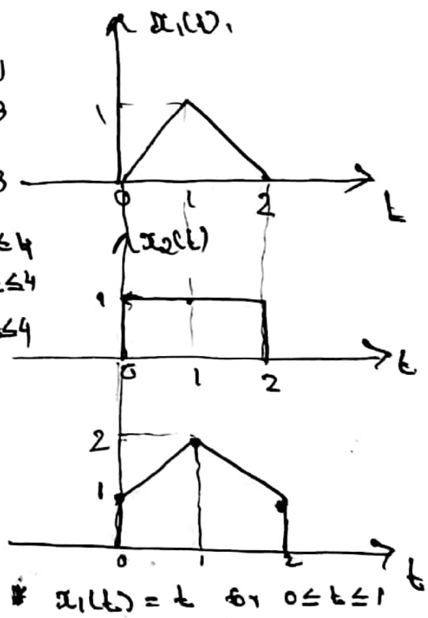




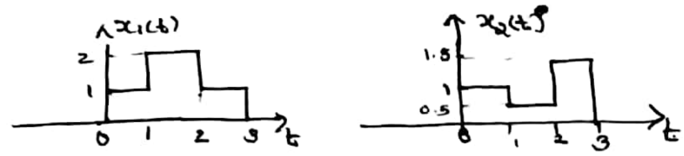
Signal addition: The sum of two continuous time signals $x_1(t)$ and $x_2(t)$ can be obtained by adding their values at every instant of time.



* $x_1(t) = 2$ for $0 \le t \le 1$
  $x_2(t) = t$ for $0 \le t \le 1$
  $y(t) = 2 + t$ for $0 \le t \le 1$
* $x_1(t) = 1$ for $1 \le t \le 3$
  $x_2(t) = 1$ for $1 \le t \le 3$
  $y(t) = 2$ for $1 \le t \le 3$
* $x_1(t) = 2$ for $3 \le t \le 4$
  $x_2(t) = -t$ for $3 \le t \le 4$
  $y(t) = 2 - t$ for $3 \le t \le 4$



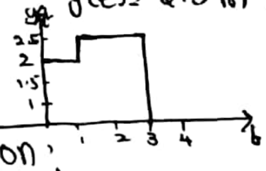
* $x_1(t) = t$ for $0 \le t \le 1$
  $x_2(t) = 1$ for $0 \le t \le 1$
  $y(t) = t + 1$ for $0 \le t \le 1$
* $x_1(t) = -t$ for $1 \le t \le 2$
  $x_2(t) = 1$ for $1 \le t \le 2$
  $y(t) = 1 - t$ for $1 \le t \le 2$



* $x_1(t) = 1$ for $0 \le t \le 1$
$x_2(t) = 1$ for $0 \le t \le 1$
$y(t) = 2$ for $0 \le t \le 1$

* $x_1(t) = 2$ for $1 \le t \le 2$
$x_2(t) = 0.5$ for $1 \le t \le 2$
$y(t) = 2.5$ for $1 \le t \le 2$

* $x_1(t) = 1$ for $2 \le t \le 3$
$x_2(t) = 1.5$ for $2 \le t \le 3$
$y(t) = 2.5$ for $2 \le t \le 3$.

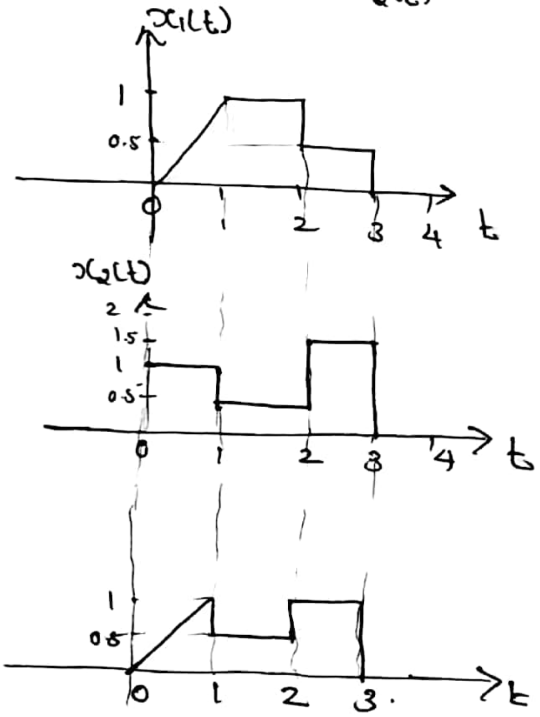


$\boxed{x(t) - x_2(t)}$

Signal multiplication:

multiplication of two signals can be obtained by multiplying their values at every instant.

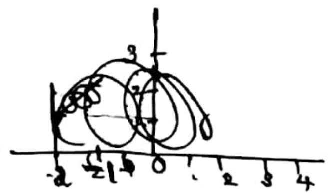
$$x_1(t) \longrightarrow \otimes \longrightarrow x_1(t) \cdot x_2(t).$$
$$\uparrow x_2(t)$$



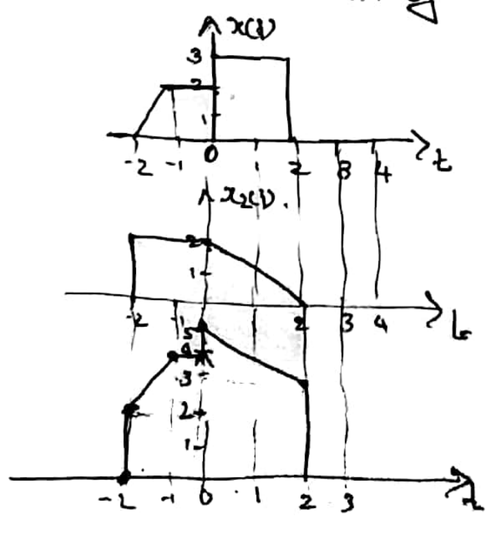
* $x_1(t) = t$ for $0 \le t \le 1$
$x_2(t) = 1$ for $0 \le t \le 1$
$y(t) = t$ for $0 \le t \le 1$

* $x_1(t) = 1$ for $1 \le t \le 2$
$x_2(t) = 0.5$ for $1 \le t \le 2$.
$y(t) = 0.5$ for $1 \le t \le 2$.

* $x_1(t) = 0.5$ for $2 \le t \le 3$
$x_2(t) = 1.5$ for $2 \le t \le 3$
$y(t) = 0.75$ for $2 \le t \le 3$.



Qn Sketch the sum of two signals.



* $x_1(t) = t \quad -2 \le t \le -1$
$x_2(t) = 2 \quad -2 \le t \le -1$
$y(t) = 2 + t \quad -2 \le t \le -1$

* $x_1(t) = 2 \quad -1 \le t < 0$
$x_2(t) = 2 \quad -1 \le t \le 0$
$y(t) = 4 \quad -1 \le t \le 0$

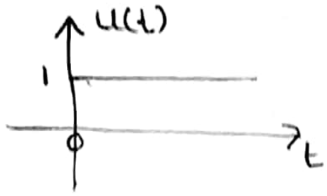
* $x_1(t) = 3 \quad 0 \le t \le 2$
$x_2(t) = -t \quad 0 \le t \le 2.$
$y(t) = 3 - t \quad 0 \le t \le 2.$

Elementary signals:

1) unit step

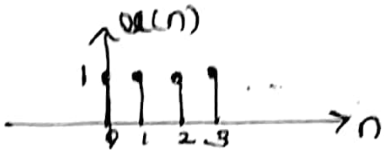
Continuous

$u(t) = 1$ for $t \geq 0$
$\quad\; = 0$ for $t < 0$



discrete

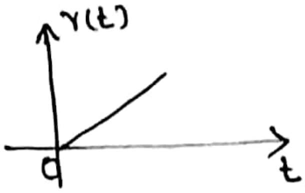
$u(n) = 1$ for $n \geq 0$
$\quad\; = 0$ for $n < 0$



2) unit ramp

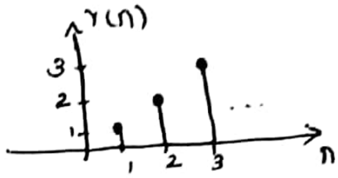
continuous

$r(t) = t$ for $t \geq 0$
$\quad\; = 0$ for $t < 0$



discrete.

$r(n) = n$ for $n \geq 0$
$\quad\; = 0$ for $n < 0$

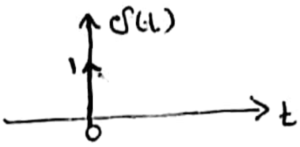


NOTE: $u(t) = \frac{d}{dt} r(t)$
(OR)
$r(t) = \int u(t)\, dt$

3) unit impulse

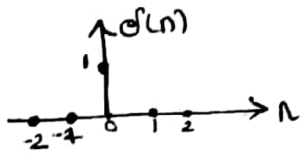
Continuous

$\delta(t) = 1$ for $t = 0$
$\quad\; = 0$ for $t \neq 0$



discrete.

$\delta(n) = 1$ for $n = 0$
$\quad\; = 0$ for $n \neq 0$



Properties:

1) $\int_{-\infty}^{\infty} x(t)\,\delta(t) = x(0)$
2) $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$
3) $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0) = x(t_0)$

1) $\delta(n) = u(n) - u(n-1)$
2) $u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$
3) $\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$

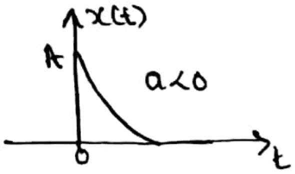
4. Exponential:

Continuous:

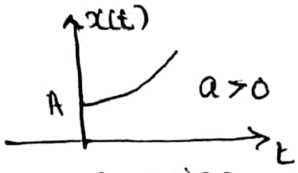
$x(t) = Ae^{at}$

~~discrete~~.

where A and a are real parameters.
The parameter A is the amplitude of the signal $x(t)$ at $t=0$.
If $a<0$, $x(t)$ is said to be decaying exponential.
If $a>0$, $x(t)$ is said to be growing exponential.



decaying

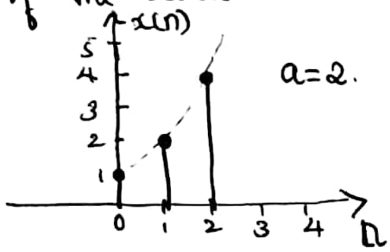


growing.

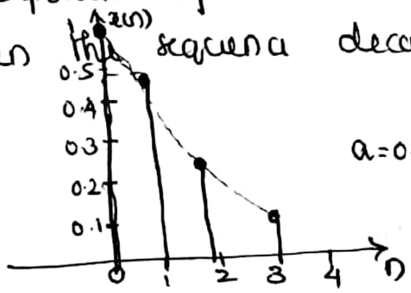
discrete:

$x(n) = a^n$ for all 'n'

If $a>1$, the sequence grows exponentially and if the value is $0<a<1$, then the sequence decay exponentially



growing



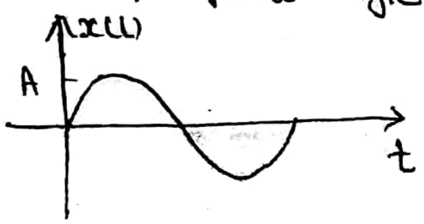
decaying.

5. Sinusoidal Signal.

Continuous

$x(t) = A\sin(\omega t+\phi)$

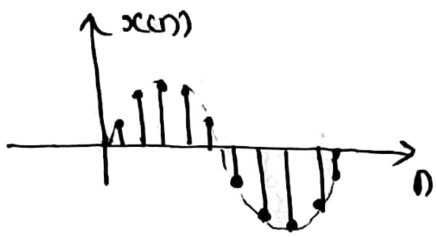
where A - amplitude of the signal.
ω - angular freq. in rad/s
ϕ - phase angle in rad.



discrete.

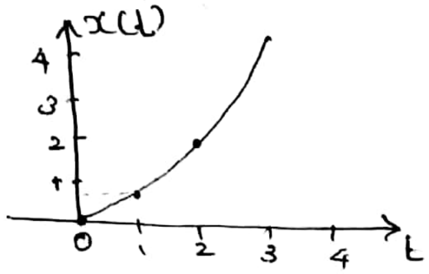
$x(n) = A\sin(\Omega n+\phi)$

Ω - angular freq.



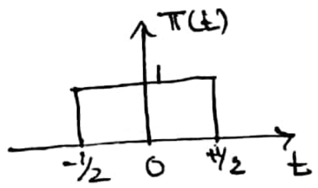
Unit parabolic function

$$p(t) = \frac{t^2}{2} \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$



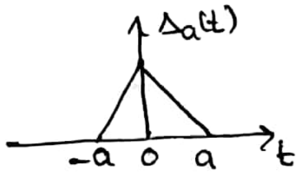
Rectangular pulse function

$$\Pi(t) = 1 \quad \text{for } |t| \leq 1/2$$
$$= 0 \quad \text{otherwise.}$$



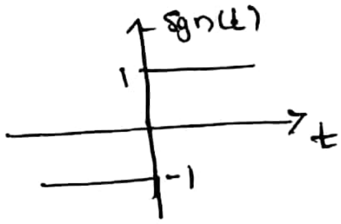
Triangular pulse function

$$\Delta_a(t) = 1 - \frac{|t|}{a} \quad |t| \leq a$$
$$= 0 \quad |t| > a.$$



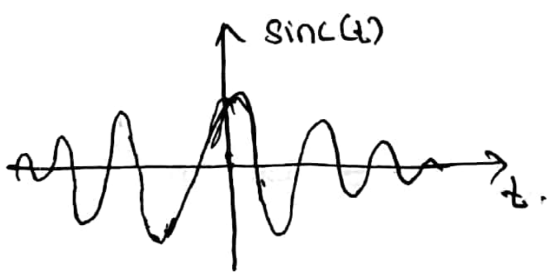
Signum function

$$\text{sgn}(t) = 1 \quad t > 0$$
$$= -1 \quad t < 0$$



Sinc function

$$\text{sinc}(t) = \frac{\sin t}{t}, \quad -\infty < t < \infty$$



Evaluate the following integrals

a) $\int_{-\infty}^{\infty} e^{-\alpha t^2} \delta(t-10)\, dt$

$$\int_{-\infty}^{\infty} x(t)\, \delta(t-10)\, dt = x(t)\Big|_{t=t_0}$$

where $x(t) = e^{-\alpha t^2}$ and $t_0 = 10$

$$\therefore \int_{-\infty}^{\infty} e^{-\alpha t^2} \delta(t-10) = e^{-\alpha t^2}\Big|_{t=10} = e^{-\alpha 10^2} = e^{-100\alpha}$$

b) $\int_{-\infty}^{\infty} t^2\, \delta(t-3) = 9$

c) $\int_{-\infty}^{\infty} [\delta(t) \cos t + \delta(t-1) \sin t]\, dt$

$$= \int_{-\infty}^{\infty} \delta(t) \cos t\, dt + \int_{-\infty}^{\infty} \delta(t-1) \sin t\, dt$$

$$= \cos t\Big|_{t=0} + \sin t\Big|_{t=1}$$

$$= \cos 0 + \sin 1 = 1 + \sin 1 //$$

Evaluate the following summations

a) $\sum_{n=-\infty}^{\infty} e^{2n} \delta(n-2)$.

$$\sum_{n=-\infty}^{\infty} x(n)\, \delta(n-n_0) = x(n)\Big|_{n=n_0}$$

where, $x(n) = e^{2n}$ and $n_0 = 2$

$$\therefore \sum_{n=-\infty}^{\infty} e^{2n} \delta(n-2) = e^{2n}\Big|_{n=2} = e^4 //$$

b) $\sum_{n=-\infty}^{\infty} \sin 2n\, \delta(n-1)$

$$= \sin 2n\Big|_{n=1} = \sin 2n.$$

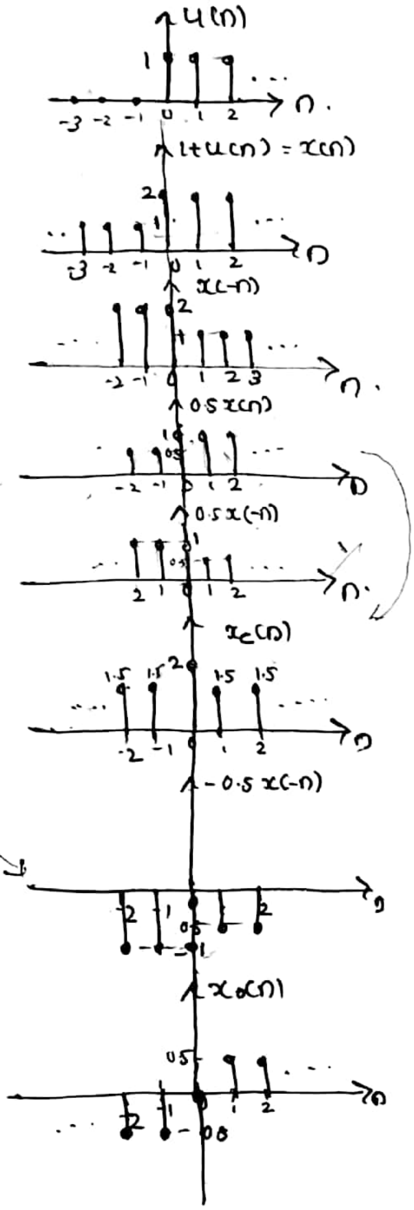
d) $\sum_{n=-\infty}^{\infty} a^{n-2} \delta(n+3)$

$$= a^{n-2}\Big|_{n=-3}$$

$$= a^{-3-2} = a^{-5} //$$

c) $\sum_{n=-\infty}^{\infty} n^2\, \delta(n+2) = n^2\Big|_{n=-2} = (-2)^2 = 4 //$

Sketch $x(n) = 1 + u(n)$ and then plot its even & odd parts.



$x_o(n) = \frac{1}{2}x(n) - \frac{1}{2}x(-n)$

Qn Find the energy of the signal $x(n) = 2^n u(-n)$

$$E = \sum_{n=-\infty}^{\infty} |2^n u(-n)|^2$$

$$= \sum_{n=-\infty}^{\infty} 4^n |u(-n)|^2 = \sum_{n=-\infty}^{0} 4^n |1|^2$$

$$= \sum_{n=-\infty}^{0} 4^n$$

$$= \sum_{n=\infty}^{0} 4^{-n} = \sum_{n=0}^{\infty} (4^{-1})^n = \sum_{n=0}^{\infty} (1/4)^n$$

$$= \frac{1}{1-1/4} = \frac{1}{\frac{4-1}{4}} = \frac{1}{3/4}$$

$$= 4/3 \text{ J}$$

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Qn. Let $x(n) = 2^n[u(n+1) - u(n-4)]$. Sketch the following signals.

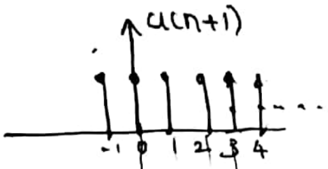
a) $y_1(n) = x(n-3)$

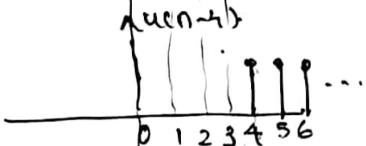
b) $y_2(n) = x(n+1)$

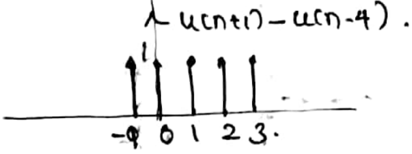
c) $y_3(n) = x(-n+4)$.

d) $y_4(n) = x(-n-2)$

Also find the energy of the signal $x(n)$.



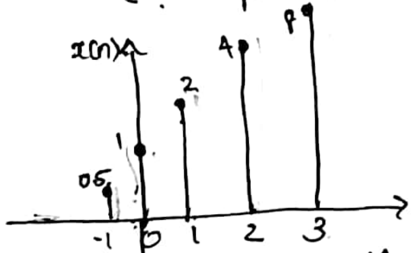


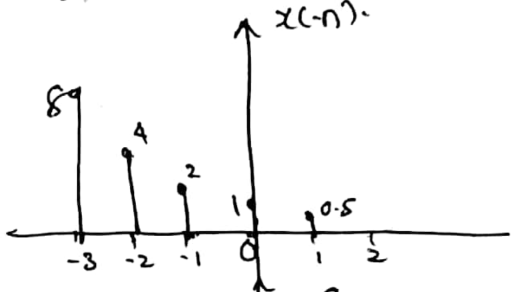


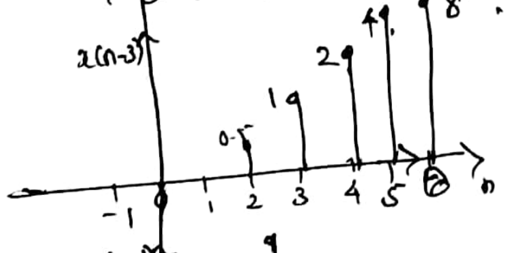
$$x(n) = 2^n\{\overset{n=-1}{1}, \underset{\uparrow}{\overset{n=0}{1}}, \overset{n=1}{1}, \overset{n=2}{1}, \overset{n=3}{1}\} = \{2^n, \underset{\uparrow}{2^n}, 2^n, 2^n, 2^n\}$$

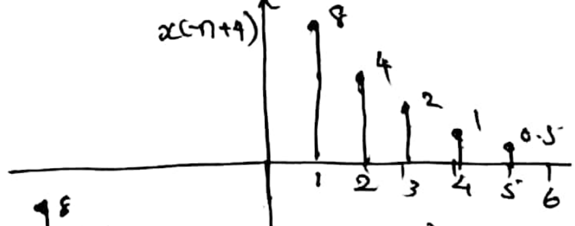
$$= (2^{-1}\cdot 1, \underset{\uparrow}{2^0\cdot 1}, 2^1\cdot 1, 2^2\cdot 1, 2^3\cdot 1)$$

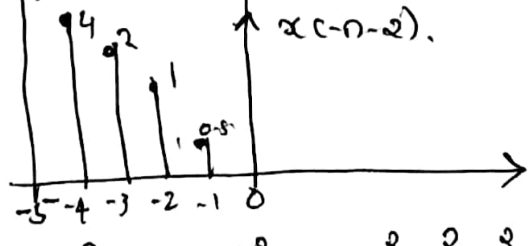
$$= \{0.5, \underset{\uparrow}{1}, 2, 4, 8\}$$











$$E = \sum_{n=-1}^{3} |x(n)|^2 = 0.5^2 + 1^2 + 2^2 + 4^2 + 8^2$$

$=$

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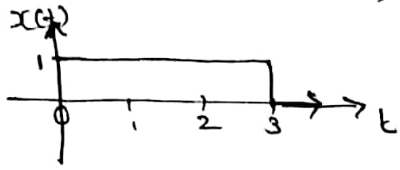
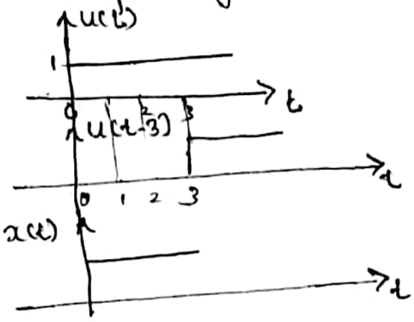
Q. Sketch the waveforms for the following signals.

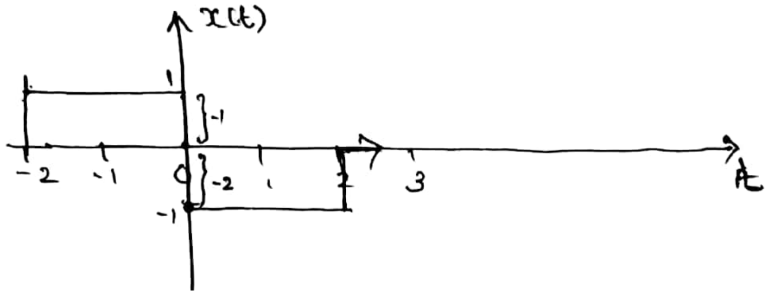
a) $x(t) = u(t) - u(t-3)$.



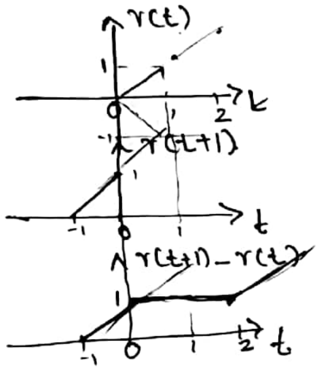


$u(t) = 1, \quad 0 \le t \le 3$
$u(t-3) = 0$
$u(t) - u(t-3) = 1 - 0 = 1$
$u(t) = 1, \quad 3 \le t \le \infty$
$u(t-3) = 1,$
$u(t) - u(t-3) = 1 - 1 = 0$

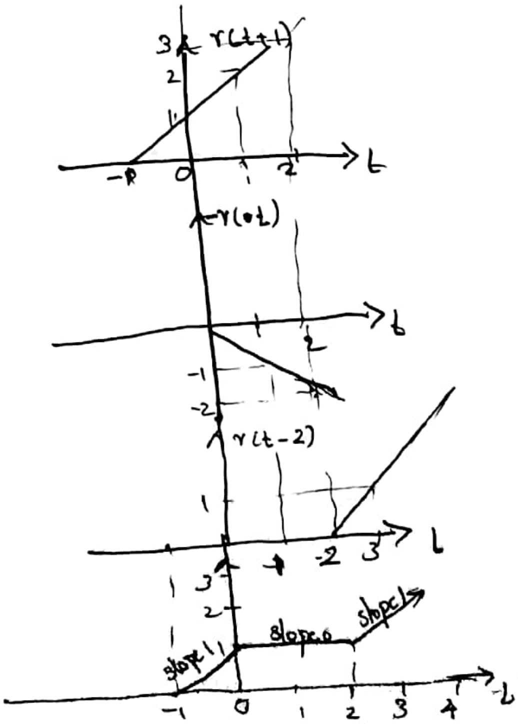
b) $x(t) = u(t+2) - 2u(t) + u(t-2)$



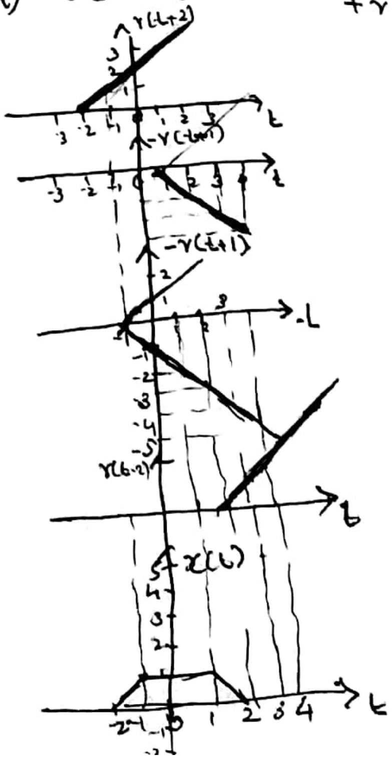
c) $r(t+1) - r(t) + r(t-2)$



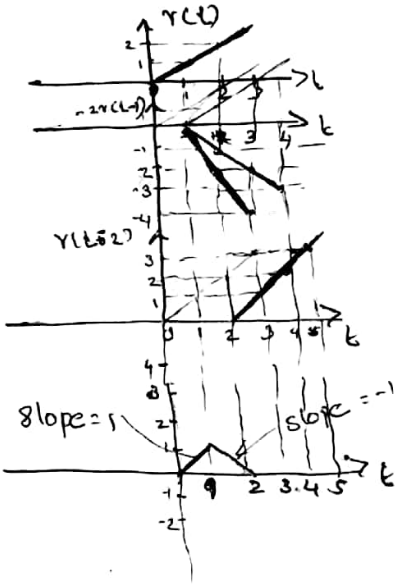
$r(t+1) = t \quad -1 \le t \le 0$
$r(t) = 0$
$r(t+1) - r(t) = t$
$r(t+1) = 1+t \quad 0 \le t \le 1$
$r(t) = t$
$r(t+1) - r(t) = 1$



d) $x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)$



e) $r(t) - 2r(t-1) + r(t-2)$

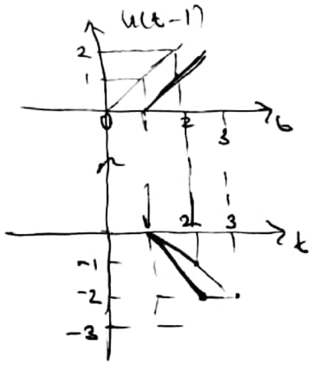


1 to 2

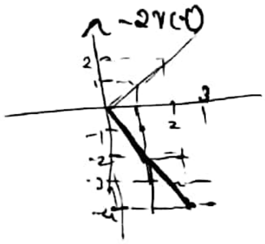
$r(t) =$ slope: 1

$-2r(t-1) = -2$

f) = $-2r(t-1)$



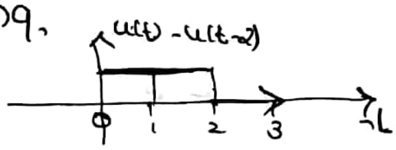
g) $-2r(t)$



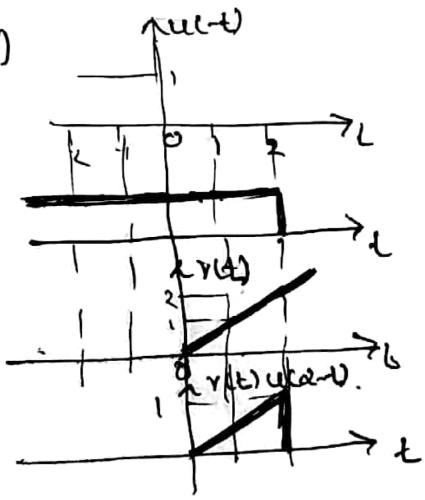
H.W

Sketch the following.

a) $u(t) - u(t-2)$



b) $r(t)\, u(2-t)$



Linear / non linear Systems:

A System that follows the Superposition theorem is said to be a linear System.

Superposition theorem states that the response to a weighted sum of input signal is equal to corresponding the weighted sum of output signal of the System

If an input $x_1(t)$ produces an output $y_1(t)$ and an i/p $x_2(t)$ produces an o/p $y_2(t)$. Then the System is linear if the weighted sum of i/p $ax_1(t)+bx_2(t)$ produces an o/p $ay_1(t)+by_2(t)$, where a & b are constants.

$$ay_1(t)+by_2(t) = T[ax_1(t)+bx_2(t)]$$

If a System does not follow the Superposition theorem or does not satisfy the eqn, then such a System is called non linear System.

||^{ly} for a discrete time System $ay_1(n)+by_2(n) = T[ax_1(n)+bx_2(n)]$

1) Check whether the following Systems are linear or non linear.

a) $y(t) = t^2 x(t)$

$y_1(t) = t^2 x_1(t)$

$y_2(t) = t^2 x_2(t)$

weighted sum of o/p : LHS: $ay_1(t)+by_2(t) = at^2x_1(t)+bt^2x_2(t)$

$= t^2[ax_1(t)+bx_2(t)]$

RHS.

$T[ax_1(t)+bx_2(t)] = t^2[ax_1(t)+bx_2(t)]$.

LHS = RHS

$\therefore$ S/m is linear.

b) $y(t) = e^{x(t)}$

$y_1(t) = e^{x_1(t)}$

$y_2(t) = e^{x_2(t)}$

LHS:
$a y_1(t) + b y_2(t) = a e^{x_1(t)} + b e^{x_2(t)}$

RHS: $T[a x_1(t) + b x_2(t)] = e^{a x_1(t) + b x_2(t)}$

LHS $\neq$ RHS $\therefore$ s/m is non linear.

c) $y(t) = \sin 6t \, x(t)$.

LHS
$a y_1(t) + b y_2(t) = a \sin 6t \, x_1(t) + b \sin 6t \, x_2(t)$

$= \sin 6t \, [a x_1(t) + b x_2(t)]$.

RHS:
$T[a x_1(t) + b x_2(t)] = \sin 6t \, [a x_1(t) + b x_2(t)]$

LHS = RHS $\therefore$ s/m is linear.

d) $y(t) = x^2(t)$.

LHS
$a y_1(t) + b y_2(t) = a x_1^2(t) + b x_2^2(t)$

RHS:

$(a x_1(t) + b x_2(t))(a x_1(t) + b x_2(t))$

$T[a x_1(t) + b x_2(t)] = x(t) \cdot x(t)$

$= [a x_1(t) + b x_2(t)]^2$

LHS $\neq$ RHS $\therefore$ non linear.

e) $y(t) = 5x(t) + 4t \, x(t-1)$.

LHS
$a y_1(t) + b y_2(t) = a 5 x_1(t) + 4at \, x_1(t-1) + 5b \, x_2(t) + 4bt \, x_2(t-1)$.

$= 5(a x_1(t) + b x_2(t)) + 4t(a x_1(t-1) + b x_2(t-1))$

RHS: $T[a x_1(t) + b x_2(t)] = 5(a x_1(t) + b x_2(t)) + 4t[a x_1(t-1) + b x_2(t-1)]$

LHS $\neq$ RHS

$\therefore$ s/m is linear.

$|x(n)| \leq m < \infty$ – Bounded i/p
$|y(n)| \leq m < \infty$ – Bounded o/p.

Stable / unstable Systems:

BIBO Stability: (Bounded i/p Bounded o/p Stability).

A system is said to be BIBO Stable, if every bounded input produces a bounded output. Mathematically $|x(n)| \leq m < \infty$, then for a BIBO stability the output $y(n)$ must obey the condition.

$|y(n)| \leq m < \infty$

For a continuous time system, if $|x(t)| \leq m < \infty$, then for a BIBO stability the o/p $y(t)$ must obey the condition $|y(t)| \leq m < \infty$.

Qn. Check whether the following S/ms are stable or not.

a) $y(n) = x(n-2)$.

$|x(n)| \leq m < \infty \rightarrow$ Bounded i/p

$|x(n-2)| \leq m < \infty$.

$|y(n)| \leq m < \infty \rightarrow$ Bounded o/p.

Bounded i/p produces Bounded o/p. ∴ S/m is stable.

b) $y(n) = x(-n)$.

$|x(n)| \leq m < \infty \longrightarrow$ Bounded i/p

$|x(-n)| \leq m < \infty$

$|y(n)| \leq m < \infty \longrightarrow$ Bounded o/p

∴ S/m is Stable.

c) $y(n) = x(n) + n$.

$|x(n)| \le m < \infty$

$|x(n) + n|$,

As $n \to \infty$ $\quad |x(n) + n| = \infty$

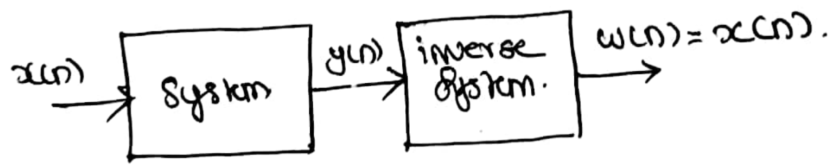
$|y(n)| = \infty$.

Bounded i/p does not produce Bounded o/p

$\therefore$ S/m is unstable.

Invertibility:

A system is said to be invertible if the input to the system can be recovered from the output. That is when the original system is cascaded with its inverse system, the output $w(n)$ is equal to the input $x(n)$ of the original system.



$y(t) = 2x(t) \Rightarrow x(t) = \frac{y(t)}{2}$ (invertible system).

$y(t) = Re\{x(t)\}$ (non invertible).

$\downarrow$

$x_1(t) = \cos t + j\sin t \quad Re\{x_1(t)\} = \cos t$

$x_2(t) = \cos t - j\sin t \quad Re\{x_2(t)\} = \cos t$.

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Qn. Determine whether the following systems are static, causal, time invariant, linear and stable.

a) $y(n) = x(4n+1)$

1. static/ dynamic:

$$y(n) = x(4n+1)$$
$$y(0) = x(1)$$

The output depends on future input $\therefore$ s/m is dynamic.

2. causal / non causal:

The output depends on future input $\therefore$ s/m is non causal.

3. time invariant / variant:

$$y(n-n_0) = x(4(n-n_0)+1)$$
$$= x(4n-4n_0+1)$$
$$T[x(n-n_0)] = x(4n+1-n_0).$$
$y(n-n_0) \neq T[x(n-n_0)]$ $\therefore$ s/m is time variant

4. linear / non linear

$$y(n) = x(4n+1)$$
$$y_1(n) = x_1(4n+1)$$
$$y_2(n) = x_2(4n+1)$$

LHS:
$$ay_1(n)+by_2(n) = ax_1(4n+1)+bx_2(4n+1)$$

RHS:
$$T[ax_1(n)+bx_2(n)] = ax_1(4n+1)+bx_2(4n+1)$$

LHS = RHS $\therefore$ s/m is linear.

5. stable/ unstable

$|x(n)| \leq m < \infty \rightarrow$ Bounded input

$\therefore |x(4n+1)| \leq m < \infty$

$|y(n)| \leq m < \infty \rightarrow$ Bounded output

Bounded i/p produces Bounded output

$\therefore$ s/m is stable.

b) $y(n) = x(n) + n\,x(n+1)$

Static / dynamic:

$y(0) = x(0)$
$y(1) = x(1) + x(2)$.

The output depends on present and future input $\therefore$ s/m is ~~dy~~ dynamic.

causal / non causal:

The o/p depends on future i/p $\therefore$ the s/m is non causal.

Time invariant / variant:

$y(n-n_0) = x(n-n_0) + (n-n_0)\,x(n-n_0+1)$

$T[x(n-n_0)] = x(n-n_0) + n\,x(n+1-n_0)$

$y(n-n_0) \neq T[x(n-n_0)]$ $\therefore$ s/m is time variant.

linear / non linear:

$y_1(n) = x_1(n) + n\,x_1(n+1)$
$y_2(n) = x_2(n) + n\,x_2(n+1)$

LHS:
$a y_1(n) + b y_2(n) = a x_1(n) + a n\,x_1(n+1) + b x_2(n) + b n\,x_2(n+1)$
$= a x_1(n) + b x_2(n) + n\,(a x_1(n+1) + b x_2(n+1))$

RHS:
$T[a x_1(n) + b x_2(n)] = a x_1(n) + b x_2(n) + n\,(a x_1(n+1) + b x_2(n+1))$

LHS = RHS $\therefore$ s/m is linear.

Stable / unstable: Bounded i/p $|x(n)| \leq m < \infty$

$y(n) = x(n) + n\,x(n+1)$.

$|y(n)| = |x(n)| + n \cdot |x(n+1)|$

As $n \to \infty$ $|y(n)| = \infty$ $\therefore$ s/m is ~~st~~ unstable.

But $y(n)$ is bounded for n = finite.

$\therefore$ s/m is stable for n = finite.

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Linearity test for the s/m described by differential equation

Step (1): write $x_1(t)$ and $x_2(t)$

Step (2): write weighted sum of i/p $ax_1(t)+bx_2(t)$ and put it as eqn no. ①

Step (3): apply the weighted sum of o/p $ay_1(t)+by_2(t)$ to the given differential eqn and put it as eqn no. ②.

Step (4): If ① = ②, then the s/m is linear otherwise non linear.

Qn. Determine whether the following differential eqns. are linear or not

a) $\frac{dy(t)}{dt} + 10y(t) = 2x(t)$. | b) $\frac{dy(t)}{dt} + 10\sin y(t) = 2x(t)$

$$x(t) = \frac{1}{2}\left[\frac{dy(t)}{dt} + 10y(t)\right]$$

1) write $x_1(t)$ & $x_2(t)$

$$x_1(t) = \frac{1}{2}\left[\frac{dy_1(t)}{dt} + 10y_1(t)\right]$$

$$x_2(t) = \frac{1}{2}\left[\frac{dy_2(t)}{dt} + 10y_2(t)\right]$$

2) write weighted sum of i/ps

$$ax_1(t)+bx_2(t) = \frac{a}{2}\left[\frac{dy_1(t)}{dt} + 10y_1(t)\right] + \frac{b}{2}\left[\frac{dy_2(t)}{dt} + 10y_2(t)\right] \longrightarrow ①$$

3) Apply the weighted sum of o/p to the given diff. eqn.

$$\frac{d}{dt}\left[ay_1(t)+by_2(t)\right] + 10\left[ay_1(t)+by_2(t)\right] = 2\left[ax_1(t)+bx_2(t)\right]$$

$$ax_1(t)+bx_2(t) = \frac{1}{2}\left[a\frac{d}{dt}y_1(t) + b\frac{d}{dt}y_2(t) + 10ay_1(t) + 10by_2(t)\right]$$

$$= \frac{a}{2}\left[\frac{dy_1(t)}{dt} + 10y_1(t)\right] + \frac{b}{2}\left[\frac{dy_2(t)}{dt} + 10y_2(t)\right] \longrightarrow ②$$

4) ① = ② ∴ s/m is linear.

b) 1) $x_1(t) = \frac{1}{2}\left[\frac{dy_1(t)}{dt} + 10\sin y_1(t)\right]$ & $x_2(t) = \frac{1}{2}\left[\frac{dy_2(t)}{dt} + 10\sin y_2(t)\right]$

2) $ax_1(t)+bx_2(t) = \frac{a}{2}\left[\frac{dy_1(t)}{dt} + 10\sin y_1(t)\right] + \frac{b}{2}\left[\frac{dy_2(t)}{dt} + 10\sin y_2(t)\right] \longrightarrow ①$

3) $\frac{d}{dt}(ay_1(t)+by_2(t)) + 10\sin(ay_1(t)+by_2(t)) = 2\left[ax_1(t)+bx_2(t)\right] \longrightarrow ②$

4) ① ≠ ② ∴ s/m is non linear.

Qn. check whether the following systems ~~are~~ is static / dynamic, linear / non linear, causal / non causal, time invariant or time variant.

a) $y(t)\frac{d^2y(t)}{dt^2} + 3t\frac{dy(t)}{dt} + y(t) = x(t)$

~~b) $\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y^2(t) = x(t)$~~

a) $y(t)\frac{d^2y(t)}{dt^2} + 3t\frac{dy(t)}{dt} + y(t) = x(t)$

i) Static / dynamic:

The system is described by differential eqn. Hence it is dynamic.

ii) Linear / non linear

(i) $x_1(t) = y_1(t)\frac{d^2y_1(t)}{dt^2} + 3t\frac{dy_1(t)}{dt} + y_1(t)$

$x_2(t) = y_2(t)\frac{d^2y_2(t)}{dt^2} + 3t\frac{dy_2(t)}{dt} + y_2(t)$

(ii) $ax_1(t) + bx_2(t) = a\left[y_1(t)\frac{d^2y_1(t)}{dt^2} + 3t\frac{dy_1(t)}{dt} + y_1(t)\right] + b\left[y_2(t)\frac{d^2y_2(t)}{dt^2} + 3t\frac{dy_2(t)}{dt} + y_2(t)\right] \longrightarrow (1)$

(iii) $(ay_1(t) + by_2(t))\frac{d^2}{dt^2}(ay_1(t) + by_2(t)) + 3t\frac{d}{dt}(ay_1(t) + by_2(t)) + (ay_1(t) + by_2(t)) = ax_1(t) + bx_2(t)$

$ax_1(t) + bx_2(t) = (ay_1(t) + by_2(t))\left(a\frac{d^2y_1(t)}{dt^2} + b\frac{d^2y_2(t)}{dt^2}\right)$

$+ 3at\frac{d}{dt}y_1(t) + 3bt\frac{d}{dt}y_2(t) + ay_1(t) + by_2(t)$

$= a^2y_1(t)\cdot\frac{d^2y_1(t)}{dt^2} + ab\,y_1(t)\frac{d^2y_2(t)}{dt^2} + ab\,y_2(t)\frac{d^2y_1(t)}{dt^2} + b^2 y_2(t)\frac{d^2y_2(t)}{dt^2}$

$+ 3at\frac{d}{dt}y_1(t) + 3bt\frac{d}{dt}y_2(t) + ay_1(t) + by_2(t)$

$= a\left[ay_1(t)\frac{d^2y_1(t)}{dt^2} + 3t\frac{dy_1(t)}{dt} + y_1(t)\right] + b\left[ay_1(t)\frac{d^2y_2(t)}{dt^2}\right.$

$\left. + a\,y_2(t)\frac{d^2y_1(t)}{dt^2} + y_2(t) + 3t\frac{dy_2(t)}{dt}\right] \longrightarrow (2)$

(1) $\neq$ (2) $\therefore$ S/m is non linear.

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3) causal / non causal

The o/p depends on the present i/p only ∴ causal

4) Time invariant / variant

The coefficient of the differential eqn are function of time.
∴ time variant.

② $5 \frac{dy(t)}{dt} + y(t) = 5x(t)$

1) dynamic

2) causal

3) Time invariant

4) $x_1(t) = \frac{dy_1(t)}{dt} + \frac{1}{5} y_1(t)$

$x_2(t) = \frac{d}{dt} y_2(t) + \frac{1}{5} y_2(t)$

$$ax_1(t) + bx_2(t) = a\left[\frac{dy_1(t)}{dt} + \frac{1}{5} y_1(t)\right] + b\left[\frac{dy_2(t)}{dt} + \frac{1}{5} y_2(t)\right] \longrightarrow ①$$

$$5 \frac{d[ay_1(t) + by_2(t)]}{dt} + ay_1(t) + by_2(t) = 5[ax_1(t) + bx_2(t)]$$

$$ax_1(t) + bx_2(t) = \frac{ad}{dt} y_1(t) + b \frac{dy_2(t)}{dt} + \frac{a}{5} y_1(t) + \frac{b}{5} y_2(t).$$

$$= a\left[\frac{dy_1(t)}{dt} + \frac{1}{5} y_1(t)\right] + b\left[\frac{dy_2(t)}{dt} + \frac{1}{5} y_2(t)\right] \longrightarrow ②$$

③ ① = ② ∴ S/m is linear.

Ans. $3\frac{dy(t)}{dt} + 5t\,y(t) = x(t)$

a) dynamic

b) causal

c) Time variant

d) $x_1(t) = 3\frac{dy_1(t)}{dt} + 5t\,y_1(t)$

$x_2(t) = 3\frac{dy_2(t)}{dt} + 5t\,y_2(t)$.

$$a x_1(t) + b x_2(t) = a\left[3\frac{dy_1(t)}{dt} + 5t\,y_1(t)\right] + b\left[3\frac{dy_2(t)}{dt} + 5t\,y_2(t)\right] \longrightarrow ①$$

$$3\frac{d}{dt}\left[a y_1(t) + b y_2(t)\right] + 5t\left[a y_1(t) + b y_2(t)\right] = a x_1(t) + b x_2(t)$$

$$a x_1(t) + b x_2(t) = 3a\frac{d y_1(t)}{dt} + 3b\frac{d}{dt}y_2(t) + 5at\,y_1(t) + 5bt\,y_2(t)$$

$$= a\left[3\frac{dy_1(t)}{dt} + 5t\,y_1(t)\right] + b\left[3\frac{dy_2(t)}{dt} + 5t\,y_2(t)\right]. \longrightarrow ②$$

① = ② $\therefore$ S/m is linear.

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* Time domain description: linear time invariant system. (LTI system)

The system that satisfies both linear and time invariant system's properties is called linear time invariant s/m.

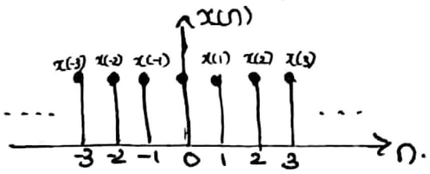
Discrete time LTI system: Convolution sum and impulse response.

when the input is impulse, then the response to the system is known as impulse response or unit sample response of the system.

$x(n) \rightarrow$ system $\rightarrow y(n)$

$x(n) \downarrow \delta(n)$ $\quad y(n) \downarrow h(n)$ — impulse response.

Consider an LTI system, the output of such a system is given by $y(n) = T[x(n)]$



$x(n) = \cdots x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots$

Let the value of impulse function at $x(0)$ is $\delta(n)$.

$x(n)$ interms of impulse function:

$$x(n) = \cdots x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \cdots$$

$$x(n) = \sum_{K=-\infty}^{\infty} x(K)\delta(n-K)$$

W.K.T $\quad y(n) = T[x(n)] = T\left[\sum_{K=-\infty}^{\infty} x(K)\delta(n-K)\right]$

$$= \sum_{K=-\infty}^{\infty} x(K)\, T[\delta(n-K)]$$

$T[x(n)] = y(n)$
$T[\delta(n)] = h(n)$
$T[\delta(n-K)] = h(n-K)$

$$= \sum_{K=-\infty}^{\infty} x(K)\, h(n-K).$$

$$= x(n) * h(n).$$

$$\therefore x(n) * h(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K). \Rightarrow \text{Convolution sum.}$$

Properties of convolution sum:

1) Commutative : $x(n) * h(n) = h(n) * x(n)$

2) Distributive : $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

3) $x(n) * \delta(n-k) = x(n-k)$.

4) $x(n) * \delta(n) = x(n)$.

5) $\delta(n-k) * \delta(n-m) = \delta(n-(m+k))$

$\delta(n) * \delta(n) = \delta(n)$

$\delta(n) * \delta(n-1) = \delta(n-1)$

Qn. Find the convolution sum of two sequences
$x_1(n) = (\underset{\uparrow}{1}, 2, 3)$ and $x_2(n) = (\underset{\uparrow}{2}, 1, 4)$

$x_1(n)$ in terms of impulse : $1 \cdot \delta(n) + 2\delta(n-1) + 3\delta(n-2)$.

$x_2(n)$ in terms of impulse : $2\delta(n) + 1\delta(n-1) + 4\delta(n-2)$

$$x_1(n) * x_2(n) = [\delta(n) + 2\delta(n-1) + 3\delta(n-2)] * [2\delta(n) + \delta(n-1) + 4\delta(n-2)]$$

$$= \delta(n) * 2\delta(n) + \delta(n) * \delta(n-1) + \delta(n) * 4\delta(n-2) + 2\delta(n-1) * 2\delta(n) + 2\delta(n-1) * \delta(n-1) + 2\delta(n-1) * 4\delta(n-2) + 3\delta(n-2) * 2\delta(n) + 3\delta(n-2) * \delta(n-1) + 3\delta(n-2) * 4\delta(n-2).$$

$$= 2\delta(n) + \delta(n-1) + 4\delta(n-2) + 4\delta(n-1) + 2\delta(n-2) + 8\delta(n-3) + 6\delta(n-2) + 3\delta(n-3) + 12\delta(n-4).$$

$$= 2\delta(n) + 5\delta(n-1) + 12\delta(n-2) + 11\delta(n-3) + 12\delta(n-4)$$

$$= (\underset{\uparrow}{2}, 5, 12, 11, 12)$$

Verification:

$x_1(n) \downarrow$ \ $x_2(n) \rightarrow$	2	1	4
1	2	1	4
2	4	2	8
3	6	3	12

$$x_1(n) * x_2(n) = (\underset{\uparrow}{2}, 5, 12, 11, 12)$$

Qn. Assume a LTI system H has impulse response

$$h(n) = \begin{cases} 1, & n = \pm 1 \\ 2, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the output of this system in response to the input

$$x(n) = \begin{cases} 2, & n = 0 \\ 3, & n = 1 \\ -2, & n = 2 \\ 0, & \text{otherwise.} \end{cases}$$

$$x(n) = (\underset{\uparrow}{2}, 3, -2) \qquad h(n) = (1, \underset{\uparrow}{2}, 1)$$

$$\begin{array}{c|ccc} x(n) \downarrow \; / \; h(n) \rightarrow & 1 & 2 & 1 \\ \hline 2 & 2 & 4 & 2 \\ 3 & 3 & 6 & 3 \\ -2 & -2 & -4 & -2 \end{array}$$

$$y(n) = x(n) * h(n) = (2, 7, 6, -1, -2)$$

$$x(n) = 2\delta(n) + 3\delta(n-1) - 2\delta(n-2)$$

$$h(n) = \delta(n+1) + 2\delta(n) + \delta(n-1).$$

$$y(n) = x(n) * h(n) = [2\delta(n) + 3\delta(n-1) - 2\delta(n-2)] * [\delta(n+1) + 2\delta(n) + \delta(n-1)]$$

$$= 2\delta(n) * \delta(n+1) + 2\delta(n) * 2\delta(n) + 2\delta(n) * \delta(n-1) +$$
$$3\delta(n-1) * \delta(n+1) + 3\delta(n-1) * 2\delta(n) + 3\delta(n-1) * \delta(n-1) +$$
$$-2\delta(n-2) * \delta(n+1) - 2\delta(n-2) * 2\delta(n) - 2\delta(n-2) * \delta(n-1)$$

$$= 2\delta(n+1) + 4\delta(n) + 2\delta(n-1) + 3\delta(n) + 6\delta(n-1) + 3\delta(n-2)$$
$$- 2\delta(n-1) - 4\delta(n-2) - 2\delta(n-3).$$

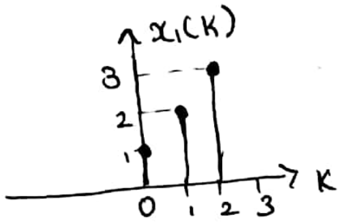
$$= 2\delta(n+1) + 7\delta(n) + 6\delta(n-1) - \delta(n-2) - 2\delta(n-3)$$

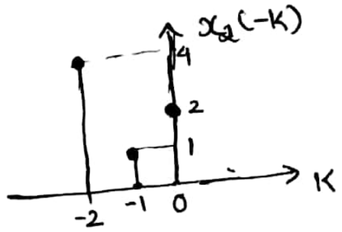
$$= (2, \underset{\uparrow}{7}, 6, -1, -2).$$

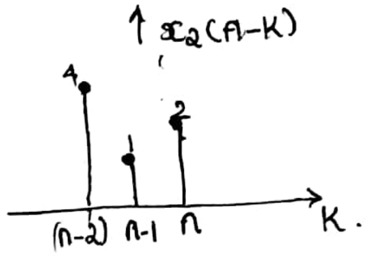
Graphical method:

$x_1(n) = (\underset{\uparrow}{1}, 2, 3)$ $\qquad$ $x_2(n) = (\underset{\uparrow}{2}, 1, 4)$

$y(n) = x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K)$. $\quad x_2(-K+n)$

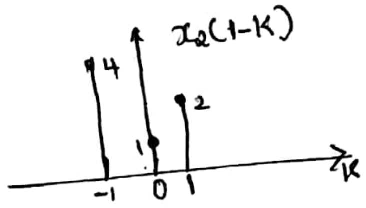






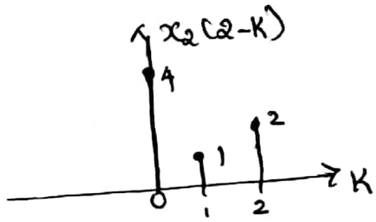
$y(0) = x_1(K) . x_2(-K) = 1 \times 2 = 2.$

$y(1) = \sum_{K=-\infty}^{\infty} x_1(K) . x_2(1-K) = 1\times1 + 2\times2 = 5$



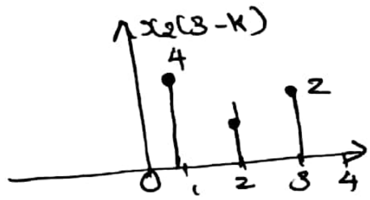
$y(2) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(2-K)$

$= 1\times4 + 2\times1 + 3\times2 = 12$



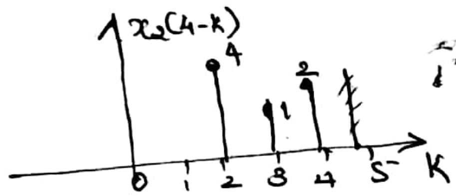
$y(3) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(3-K)$

$= 2\times4 + 3\times1 = 11$



$y(4) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(4-K)$

$= 3\times4 = 12.$



$y(5) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(5-K)$

$= 0$

$\therefore y(n) = (\underset{\uparrow}{2}, 5, 12, 11, 12)$

Graphical method: (Steps)

~~Step~~ $x_1(n) = (\underset{\uparrow}{1}, 2, 3)$ $\quad x_2(n) = (\underset{\uparrow}{2}, 1, 4)$

Step (1): $x_1(n)$: Starting point 0
$x_2(n)$: starting point 0
$y(n)$: Starting point $0+0 = 0$

Step (2): $\text{length}(x_1(n)) \overset{(N_1)}{=} 3$
$\text{length}(x_2(n)) \underset{(N_2)}{=} 3$
$\therefore \text{Length}(y(n)) = N_1 + N_2 - 1 = 6 - 1 = 5$.

Step (3): Express ~~the~~ $x_1(n)$ & $x_2(n)$ and $x_2(-n)$ in terms of K.

Step (4): write the convoln. sum. eqn:

$$x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K).$$

$$y(0) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(-K).$$

$$y(1) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(1-K)$$

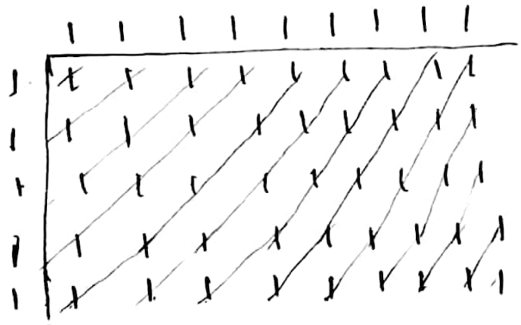
$$y(2) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(2-K)$$

$$y(3) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(3-K)$$

$$y(4) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(4-K).$$

Step (5): plot $y(n)$.

Qn. A LTI System has impulse response given by $h(n) = u(n) - u(n-10)$. Determine the output of this System when the input is the rectangular pulse defined as $x(n) = u(n-2) - u(n-7)$.

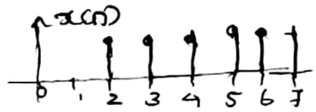


0, 2, 3 - 9, 10 -
10 ...

h(n): 0, 2 ... 9

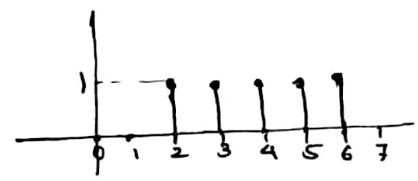
2, 3, 4 . 7 ... ∞
7 ... ∞

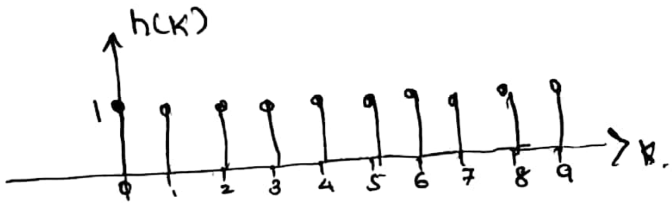
x(n): (2, 3, 4, 5, 6)

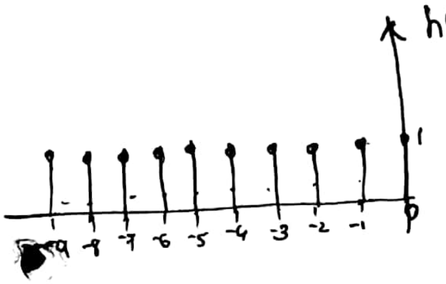


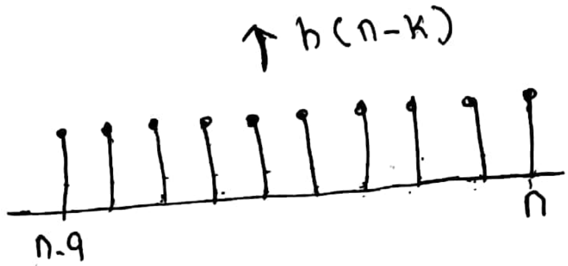
$$y(n) = \{1, 2, 3, 4, 5, 5, 5, 5, 5, 4, 3, 2, 1\}$$

2 3 4 5 6 7 8 9 10 11 12 13 14







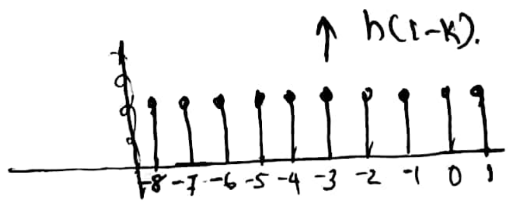


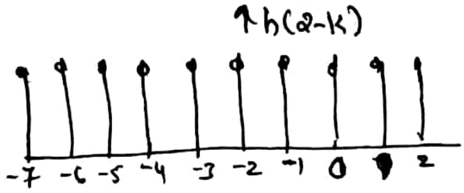
$$y(0) = \sum_{K=-\infty}^{\infty} x(K).\, h(-K) = 0$$

$$y(1) = \sum_{K=-\infty}^{\infty} x(K)\, h(1-K) = 0.$$

$$y(2) = \sum_{K=-\infty}^{\infty} x(K)\, h(2-K) = 1$$

$$y(3) = \sum_{K=-\infty}^{\infty} x(K)\, h(3-K) = 2.$$





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Continuous time LTI system: Convolution Integral of impulse response.

Input interms of impulse can be written by integral form as follows:

$$x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

W.K.T

$$y(t) = T[x(t)] = T\left[\int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\,T\delta(t-\tau)\,d\tau = \int_{\tau=-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= x(t) * h(t).$$

$$\therefore x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \rightarrow \text{Convolution Integral}$$

Properties of Convolution Integral:

Commutative:

1) $x(t) * h(t) = h(t) * x(t)$

2) Distributive :

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$

3) $x(t) * \delta(t-k) = x(t-k)$.

Qn. Find the convolution of $x_1(t) = e^{-at}u(t)$ and $x_2(t) = e^{-bt}u(t)$.

$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau.$$

$$= \int_{\tau=-\infty}^{\infty} e^{-a\tau} u(\tau)\; e^{-b(t-\tau)} u(t-\tau)\, d\tau .$$

$t - \tau \geq 0$

$t \geq \tau$

$0 \leq \tau \leq t$

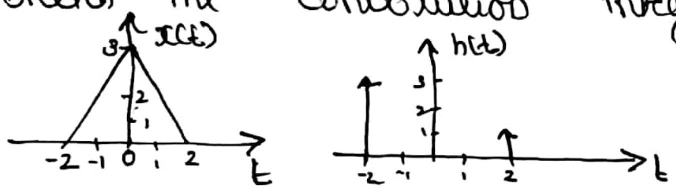
$$= \int_{\tau=0}^{t} e^{-a\tau}\, e^{-bt}\, e^{b\tau}\, d\tau .$$

$$= e^{-bt} \int_{\tau=0}^{t} e^{-a\tau} e^{b\tau}\, d\tau = e^{-bt} \int_{\tau=0}^{t} e^{-(a-b)\tau}\, d\tau .$$

$$= e^{-bt} \left[\frac{e^{-(a-b)\tau}}{-(a-b)} \right]_0^t = \frac{e^{-bt}}{-(a-b)} \left[e^{-(a-b)t} - e^{0} \right] .$$

$$= \frac{e^{-bt}}{(b-a)} \left[e^{-at} \cdot e^{bt} - 1 \right] = \frac{e^{-at} - e^{-bt}}{b-a} .$$

Qn. Sketch the convolution integral



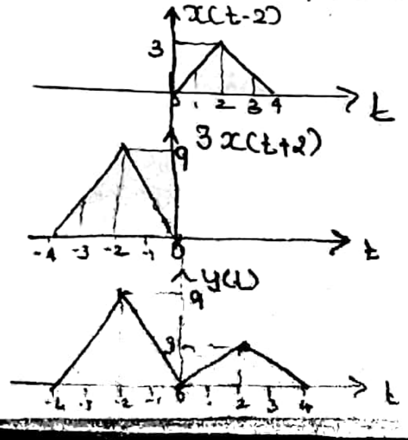
$h(t) = 1 \cdot \delta(t-2) + 3\delta(t+2)$.

in terms of impulse

$y(t) = x(t) * h(t) = x(t) * [3\delta(t+2) + \delta(t-2)]$

$= x(t) * 3\delta(t+2) + x(t) * \delta(t-2)$

$= 3\, x(t+2) + x(t-2)$.



Qn. Find ~~the~~ $y(t)$ if $x(t) = u(t)$ and $h(t) = u(t)$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau$$

$t-\tau \geq 0$
$t \geq \tau$
$\tau \leq t$

$$= \int_0^t d\tau = [\tau]_0^t = [t-0]$$

$$= t \cdot u(t)$$

Qn. Find $y(t)$ if $x(t) = e^{at}u(t)$ and $h(t) = u(t)$

$$y(t) = \int_{\tau=-\infty}^{\infty} e^{a\tau} u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{\tau=0}^{t} e^{a\tau}\, d\tau = \left[\frac{e^{a\tau}}{a}\right]_0^t$$

$$= \frac{e^{at}}{a} - \frac{1}{a} = \frac{1}{a}\left[e^{at} - 1\right] u(t)$$

Qn. Find $y(t)$ if $x(t) = e^{-3t}u(t)$ and $h(t) = (2 - e^{-2t})\, u(t)$

$$y(t) = \int_{\tau=-\infty}^{\infty} e^{-3\tau} u(\tau) \left[2 - e^{-2(t-\tau)}\right] u(t-\tau)\, d\tau$$

$$= \int_{\tau=0}^{t} \left[2e^{-3\tau} - e^{-3\tau} e^{-3\tau - 2t} - e^{-3\tau} e^{2\tau}\right] d\tau$$

$u(t) = \left[\frac{2}{3} + \frac{1}{3}e^{-3t} - e^{-2t}\right] u(t)$

$$= \int_{\tau=0}^{t} \left[2e^{-3\tau} - e^{-(3\tau+2t)} - e^{-\tau}\right] d\tau$$

$$= \left[-\frac{2}{3}e^{-3\tau} - \cancel{\frac{e^{-(3\tau+2t)}}{-(3\tau+2t)}} + e^{-\tau}\right]_0^t$$

$\frac{e^{-3\tau} e^{2t}}{-3}$

$$= \left[-\frac{2}{3}e^{-3t} + \frac{e^{2t}}{3} e^{-3t} + e^{-t} + \frac{2}{3} - \frac{e^{2t}}{3} - 1\right]$$

$$= \frac{2}{3} - \frac{2}{3}e^{-3t} + \frac{e^{-t}}{3} + e^{-t} - \frac{e^{2t}}{3}$$

Qn Find the convolution of $x(t) = t$ and $h(t) = u(t)$

Qn. Find the convolution of $x(t) = \sin at\, u(t)$ & $h(t) = u(t)$

$$y(t) = \int_{\tau=-\infty}^{\infty} \sin a\tau\, u(\tau) \cdot u(t-\tau)\, d\tau$$

$$= \int_{\tau=0}^{t} \sin a\tau\, d\tau = \left[-\frac{\cos a\tau}{a}\right]_0^t$$

$$= \frac{1}{a}\left[-\cos at + 1\right]$$

$$= \frac{1}{a}\left[1 - \cos at\right] u(t).$$

Qn. Solve for $y(t)$ if $h(t) = e^{at} u(t)$ and
$x(t) = e^{bt} u(-t)$

For $t < 0$

$$y_1(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

$$= \int_{-\infty}^{\infty} e^{b\tau} u(-\tau)\, e^{a(t-\tau)} u(t-\tau)\, d\tau.$$

$$= \int_{-\infty}^{t} e^{b\tau} e^{a(t-\tau)}\, d\tau.$$

$$= e^{at} \int_{-\infty}^{t} e^{(b-a)\tau}\, d\tau = e^{at}\left[\frac{e^{(b-a)\tau}}{b-a}\right]_{-\infty}^{t}$$

$$= \frac{e^{at}}{b-a}\left[e^{(b-a)t} - \frac{1}{b-a}\right]$$

if $b > a$ for convergence.

$$= \frac{e^{at}}{b-a}\left[e^{(b-a)t}\right] u(-t).$$

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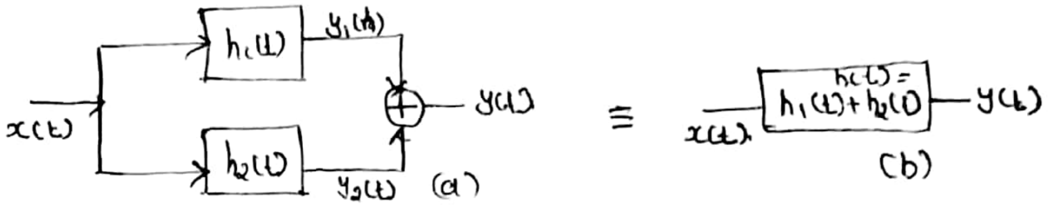
Representation of LTI Systems:
* Parallel
* cascade

1) Parallel connection of two systems:

Consider two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ connected in parallel as shown in fig. given below.



From the fig (a), $y(t) = y_1(t) + y_2(t)$

$$= x(t) * h_1(t) + x(t) * h_2(t).$$

Substitute the integral representation for each convolution

ie,

$$y_1(t) = x(t) * h_1(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau$$

$$y_2(t) = x(t) * h_2(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h_2(t-\tau)\, d\tau$$

$$\therefore y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau + \int_{\tau=-\infty}^{\infty} x(\tau)\, h_2(t-\tau)\, d\tau.$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, [h_1(t-\tau) + h_2(t-\tau)]\, d\tau.$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau).\, d\tau.$$

$$= x(t) * h(t) = x(t) * [h_1(t) + h_2(t)]$$

$$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)]$$

Ill'y for discrete time signals.

$$x(n) * h_1(n) + x(n) * h_2(n) = x(n) * [h_1(n) + h_2(n)].$$

$\therefore$ The impulse response of two systems connected in parallel is the sum of individual impulse responses.

Proof: $x(n) * h_1(n) + x(n) * h_2(n)$

$$= \sum_{K=-\infty}^{\infty} x(K)\, h_1(n-K) + \sum_{K=-\infty}^{\infty} x(K)\, h_2(n-K)$$

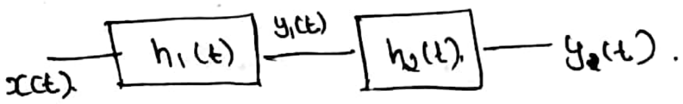
$$= \sum_{K=-\infty}^{\infty} x(K) \left[h_1(n-K) + h_2(n-K)\right]$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, h(n-K) = x(n) * h(n). \quad \text{where } h(n) = h_1(n) + h_2(n)$$

$$= x(n) * \left[h_1(n) + h_2(n)\right].$$

2) Cascade connection of two systems:

Consider two systems connected in cascade with their impulse responses $h_1(t)$ and $h_2(t)$ as shown in fig given below.



o/p of 1st s/m: $y_1(t) = x(t) * h_1(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau$

o/p of 2nd s/m: $y(t) = y_1(t) * h_2(t) = \int_{m=-\infty}^{\infty} y_1(m)\, h_2(t-m)\, dm$

$$= \int_{m=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x(\tau)\, h_1(m-\tau)\, d\tau\right] h_2(t-m)\, dm$$

Put $P = m - \tau \Rightarrow m = P + \tau$, $dm = dP$.

Changing the order of I'ion.

$$\therefore y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{P=-\infty}^{\infty} h_1(P)\, h_2(t-(P+\tau))\, dP\right] d\tau.$$

$\to h_1(t) * h_2(t-\tau) = h(t-\tau)$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{P=-\infty}^{\infty} \underbrace{h_1(P)\, h_2((t-\tau)-P)\, dP}_{h(t-\tau)}\right] d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t).$$

$$= x(t) * [h_1(t) * h_2(t)]$$

$$\boxed{x(t) * h_1(t) * h_2(t) = x(t) * [h_1(t) * h_2(t)]}$$

IIIly for discrete time signals.

$$\underbrace{x(n) * h_1(n)}_{y_1(n)} * h_2(n) = x(n) * [h_1(n) + h_2(n)]$$

Proof:

$$x(n) \longrightarrow \boxed{h_1(n)} \xrightarrow{y_1(n)} \boxed{h_2(n)} \longrightarrow y(n)$$

$$y_1(n) = x(n) * h_1(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K).$$

$$y(n) = y_1(n) * h_2(n) = \sum_{m=-\infty}^{\infty} y_1(m)\, h_2(n-m).$$

$$= \sum_{m=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x(K)\, h_1(m-K)\right] h_2(n-m)$$

Put $P = m-K \Rightarrow m = P+K$.

$$\therefore y(n) = \sum_{K=-\infty}^{\infty} x(K) \sum_{P=-\infty}^{\infty} h_1(P)\, h_2(n-(P+K))$$

$$= \sum_{K=-\infty}^{\infty} x(K) \underbrace{\sum_{P=-\infty}^{\infty} h_1(P)\, h_2((n-K)-P)}_{h_1(n-K) * h_2(n-K) = h(n-K)}$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, h(n-K).$$

$$= x(n) * h(n) = x(n) * [h_1(n) * h_2(n)]$$

The impulse response of a cascade connection of LTI systems is given by the convolution of the individual impulse responses.

Properties of LTI Systems:

1) Commutative property:

$$x(t) * h(t) = h(t) * x(t)$$

Proof: $x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

Put $t-\tau = p \Rightarrow \tau = t-p$

$\frac{dp}{d\tau} = -1 \quad d\tau = -dp$

$\therefore \; x(t) * h(t) = -\int_{p=\infty}^{-\infty} x(t-p)\, h(p)\, dp = \int_{p=-\infty}^{\infty} h(p)\, x(t-p)\, dp$

$= h(t) * x(t)$.

2) Distributive property: Reg $x_0(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2$

Proof: Refer parallel connection of two system

3) Associative property: $x(t) * h_1(t) * h_2(t) = x(t) * [h_1(t) * h_2(t)]$

proof: Refer cascade connection of two system.

4) Systems with and without memory:

Static (or memoryless): o/p at any time depends only on the value of i/p at that time.

A continuous time LTI s/m is memoryless if $h(t) = 0$ for $t \neq 0$ and such a system has the form: $h(t) = K\delta(t)$, $\therefore h(t) \neq 0$ for $t \neq 0$

5) Causal system:

causal: $h(t) = 0$ for $t < 0$

non causal: $h(t) \neq 0$ for $t < 0$

Properties of impulse response

The impulse response completely characterizes the input-o/p behaviour of a LTI system. Hence properties of a system, such as memory, causality and stability, are related to its impulse response. memoryless s/m: The o/p of a memoryless s/m depends only on the present i/p. The output of a LTI discrete time system may be expressed as $y(n) = x(n) * h(n) = h(n) * x(n)$

$$= \sum_{K=-\infty}^{\infty} h(K)\, x(n-K).$$

For the system to be memoryless, $y(n)$ must depend only on $x(n)$ and cannot depend on $x(n-K)$ for $K \neq 0$. This condition implies that $h(K) = 0$ for $K \neq 0$. Hence LTI system is memoryless if and only if $h(K) = c\delta(K)$, where c – arbitrary constant. For continuous time systems, $h(\tau) = c\delta(\tau)$.

Causal system: The output of a causal system depends only on past or present values of the i/p. $y(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$.
In order for $y(n)$ to depend only on past or present values of the input, we require $h(K) = 0$ for $K<0$. Hence for a causal s/m, $h(K) = 0$ for $K<0$ and the convoln. sum is rewritten as

$$y(n) = \sum_{K=0}^{\infty} h(K)\, x(n-K)$$

ll^{ly} For a causal continuous time s/m, $h(\tau) = 0$ for $\tau < 0$ and the o/p of a causal system is thus expressed as the convoln. Integral

$$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau.$$

Stability in terms of impulse response:

According to BIBO stability criteria, for a system to be stable, it has to produce bounded o/p for a bounded i/p. Let us consider an i/p $x(t)$ that has a bounded magnitude $|x(t)| \leq m < \infty$

$y(t) = T[x(t)]$

W.K.T $y(t) = x(t) * h(t) = h(t) * x(t)$. (Commutative property)

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau.$$

$$= \int_{-\infty}^{\infty} |h(\tau)| \cdot m \, d\tau.$$

$|y(t)| \leq m < \infty$.

$$\int_{-\infty}^{\infty} |h(\tau)| \cdot m \, d\tau \leq m < \infty$$

$$\therefore \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

$\therefore$ the s/m is stable if the impulse response is absolutely integrable.

$|||^{ly}$ for a discrete time s/m, $\sum_{K=-\infty}^{\infty} |h(K)| < \infty$. ie the s/m is stable if the impulse response is absolutely summable.

Invertible System:

$$x(t) * [h(t) * h^{-1}(t)] = x(t) * \delta(t) = x(t)$$

$$\text{ie} \quad h(t) * h^{-1}(t) = \delta(t)$$

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Differential and Difference equation representations for LTI System:

Differential equations are used to represent continuous time systems and difference eqns. are used to represent discrete time systems.

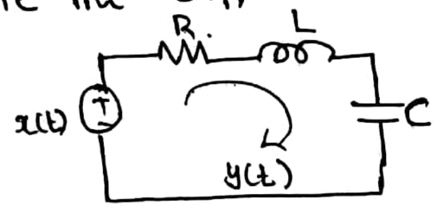
The general form of a linear constant - coefficient differential eqn. is $\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$

where $x(t)$ is the i/p to the system and $y(t)$ is the output.

The general form of a linear constant coefficient difference eqn is $\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$.

The integer N is termed as the order of the differential or difference eqn. and correspond to the highest derivative involving the system output.

Qn. Write the differential eqn. for the system given below.



$$R\,y(t) + L\frac{dy(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t).$$

differentiating both sides w.r.to t

$$R\frac{dy(t)}{dt} + L\frac{d^2y(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}$$

order, $N = 2$//.

Qn Give an example of a second order difference eqn.

$$y(n) + y(n-1) + \frac{1}{4} y(n-2) = x(n) + 2x(n-1).$$

order, $N = 2$

Total response = zero i/p response + zero state response.

(natural response) + (Forced response)
$(y_n(n))$

Solution of homogeneous eqn. $(y_h(t))$; $y_h(t) + y_p(t)$ → Particular soln.

<u>Continuous time.</u>

i/p	Particular soln.
1	K
e^{-at}	Ke^{-at}
$\cos(\omega t)$	$K_1 \cos \omega t + K_2 \sin \omega t$

<u>Discrete time</u>

i/p	particular soln.
1	K.
α^n	$K\alpha^n$.
$\cos \Omega n$	$K_1 \cos \Omega n + K_2 \sin \Omega n$

Qn. Find the total response of the system described by the differential eqn: $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$

When input is $x(t) = e^{-t}u(t)$ and the initial condition is $y(0) = 3$, $\frac{dy(0)}{dt} = 0$.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t) \rightarrow (1)$$

<u>Natural response:</u>

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 0 \rightarrow (2)$$

characteristic eqn is $\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+2)(\lambda+3) = 0$

$\lambda_1 = -2$, $\lambda_2 = -3$

homogeneous eqn: $y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 (t)} + \cdots$

$$y_h(t) = c_1 e^{-2t} + c_2 e^{-3t} \longrightarrow ③$$

Soln: $y_h(t)\Big|_{t=0} = c_1 + c_2 = 3 \longrightarrow 3(a)$

$$\frac{dy(t)}{dt} = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

$$\frac{dy(t)}{dt}\Big|_{t=0} = -2c_1 - 3c_2 = 0 \longrightarrow 3(b).$$

Solving 3(a) & 3(b), we get

$$2c_1 + 2c_2 = 6$$
$$-2c_1 - 3c_2 = 0$$
$$-c_2 = 6 \Rightarrow c_2 = -6$$

$\therefore c_1 = 9.$

natural resp:

$y_n(t) =$ soln. of $y_h(t)$

$$y_n(t) = 9e^{-2t} - 6e^{-3t} \longrightarrow ④$$

Forced response:

For i/p e^{-t}, the particular soln is

$$\left.\begin{aligned} y_p(t) &= Ke^{-t} \\ \frac{dy_p(t)}{dt} &= -Ke^{-t} \\ \frac{d^2y_p(t)}{dt} &= Ke^{-t} \end{aligned}\right\} ⑤$$

particular soln. should satisfy the differential eqn.

Sub eqn ⑤ in ①

$$\Rightarrow \frac{d^2y_p(t)}{dt^2} + 5\frac{dy_p(t)}{dt} + 6y_p(t) = \frac{dx(t)}{dt} + 4x(t)$$

$x(t) = e^{-t}$

$\frac{dx(t)}{dt} = -e^{-t}$

$$Ke^{-t} - 5Ke^{-t} + 6Ke^{-t} = -e^{-t} + 4e^{-t}$$

$$2K\cancel{e^{-t}} = 3\cancel{e^{-t}}$$

$$K = 3/2 = 1.5.$$

$$\therefore y_p(t) = 1.5e^{-t} \longrightarrow ⑥$$

Forced response = homogeneous eqn + particular soln.

$$y_f(t) = c_1 e^{-2t} + c_2 e^{-3t} + 1.5e^{-t} \longrightarrow ⑦$$

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$y_f(t)\big|_{t=0} = c_1 + c_2 + 1.5 = 0$ (∵ zero state response) $\rightarrow$ 8(a)

$$\frac{dy_f(t)}{dt} = -2c_1 e^{-2t} - 3c_2 e^{-3t} - 1.5 e^{-t}$$

$$\frac{dy_f(t)}{dt}\bigg|_{t=0} = -2c_1 - 3c_2 - 1.5 = 0. \longrightarrow 8(b)$$

Solving 8(a) and 8(b), we get $c_2 = 1.5$ & $c_1 = -3$

$\therefore y_f(t) = -3e^{-2t} + 1.5e^{-3t} + 1.5e^{-t}$

$\therefore$ Total response = $y_n(t) + y_f(t)$

$$y(t) = (9e^{-2t} - 6e^{-3t}) + -3e^{-2t} + 1.5e^{-3t} + 1.5e^{-t}$$

$$y(t) = \underline{6e^{-2t} - 4.5e^{-3t} + 1.5e^{-t}}$$

Qn. By using the classical method, solve

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t)$$

if the initial conditions are $y(0^+) = \frac{9}{4}$, $\frac{dy(0)}{dy} = 5$ and if the input is $e^{-3t}u(t)$.

Repeated roots $y_h(t) = (c_1 + c_2 t + c_3 t^2 + \ldots)e^{\lambda t}$

$$\frac{d^2y(t)}{dt} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t) \longrightarrow (1)$$

natural resp:

charac. eqn: $\lambda^2 + 4\lambda + 4 = 0$.

$(\lambda+2)^2 = 0 \quad \lambda = \underbrace{-2, -2}$ Repeated roots

homogeneous eqn: $y_h(t) = (c_1 + c_2 t)e^{\lambda t}$

$$y_h(t) = (c_1 + c_2 t)e^{-2t} \longrightarrow (2)$$

Soln: $y_h(t)\big|_{t=0} = c_1 = 9/4$

$$\frac{dy_h(t)}{dt} = -2c_1 e^{-2t} + c_2\left[-t\,2e^{-2t} + e^{-2t}\cdot 1\right]$$

$\frac{d}{dt}u.v = u.\frac{dv}{dt} + v\frac{du}{dt}$

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$$\left.\frac{dy_n(t)}{dt}\right|_{t=0} = -2c_1 + c_2 = 5$$

$$= -\cancel{2} \times \frac{9}{4_2} + c_2 = 5$$

$$= -\frac{9}{2} + c_2 = 5 \Rightarrow c_2 = 5 + \frac{9}{2} = \frac{10+9}{2} = \frac{19}{2}$$

$$\therefore y_n(t) = \left(9/4 + \frac{19}{2} t\right) e^{-2t} \longrightarrow (3)$$

<u>Forced response:</u>

For the input $\cancel{e^{-3}}\, x(t) = e^{-3t} u(t)$, the particular soln. is

$$y_p(t) = K e^{-3t} \cancel{u(t)}$$

$$\frac{dy_p(t)}{dt} = -3Ke^{-3t}$$

$$\frac{d^2 y_p(t)}{dt} = 9Ke^{-3t}$$

$$(1) \Rightarrow \frac{d^2 y_p(t)}{dt} + 4\frac{dy_p(t)}{dt} + 4y_p(t) = \frac{dx(t)}{dt} + x(t) \quad \Bigg| \quad x(t) = e^{-3t}, \quad \frac{dx(t)}{dt} = -3e^{-3t}$$

$$9Ke^{-3t} + 4(-3Ke^{-3t}) + 4Ke^{-3t} = -3e^{-3t} + e^{-3t}$$

$$9Ke^{-3t} - 12Ke^{-3t} + 4Ke^{-3t} = -3e^{-3t} + e^{-3t}$$

$$K\cancel{e^{-3t}} = \cancel{e^{-3t}}(-3+1) \Rightarrow K = -2,$$

$$\therefore y_p(t) = -2e^{-3t} \longrightarrow (4)$$

Forced resp, $y_f(t) = y_h(t) + y_p(t)$

(zero state resp)

$$= (c_1 + c_2 t)e^{-2t} + -2e^{-3t} \longrightarrow (5)$$

$$\left. d_p(t) \right|_{t=0} = c_1 \cancel{e^{-2t}} - 2 = 0 \Rightarrow c_1 = 2$$

$$\frac{d_f(t)}{dt} = -2c_1 e^{-2t} + c_2\left[t \cdot 2e^{-2t} + e^{-2t} \cdot 1\right] + 6e^{-3t}$$

$$\left.\frac{d_p(t)}{dt}\right|_{t=0} = -2c_1 + c_2 + 6 = 0 \Rightarrow -2\times 2 + c_2 + 6 = 0$$

$$-4 + c_2 + 6 = 0$$

$$\Rightarrow c_2 = \quad c_2 + 2 = 0$$

$$c_2 = -2$$

$y_p(t) = (2 - 2t)e^{-2t} - 2e^{-3t}$

Total resp: $y(t) = y_n(t) + y_p(t)$

$$= \left(\frac{9}{4} + \frac{19}{2}t\right)e^{-2t} + (2-2t)e^{-2t} - 2e^{-3t}$$

$$= \frac{9}{4}e^{-2t} + \frac{19}{2}te^{-2t} + 2e^{-2t} - 2te^{-2t} - 2e^{-3t}$$

$$= \frac{17}{4}e^{-2t} + \frac{15}{2}te^{-2t} \cancel{- 2te^{-2t}} - 2e^{-3t}$$

$\frac{9}{4} + 2 = \frac{9+8}{4} = \frac{17}{4}$

$\frac{19}{2} - 2$

$\frac{19-4}{2} = \frac{15}{2}$

$=$

$\frac{15}{2} - 1 = \frac{15-2}{2}$

$= 13/2$

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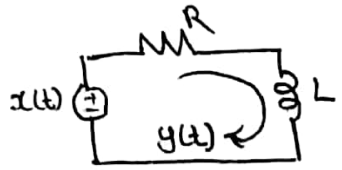
Qn. Find the current in the RL circuit given below for an applied voltage $x(t) = \cos t$ volts assuming normalized values $R = 1\Omega$, $L = 1H$ and the initial condition, $y(0) = 2A$



$$Ry(t) + L\frac{dy(t)}{dt} = x(t).$$

$$y(t) + \frac{dy(t)}{dt} = x(t). \longrightarrow ①$$

natural resp:

$$\lambda + 1 = 0 \Rightarrow \lambda = -1$$

homogeneous eqn: $y_h(t) = C_1 e^{\lambda t} \Rightarrow y_h(t) = C_1 e^{-t} \longrightarrow ②$

soln: $y_h(t)\big|_{t=0} = C_1 = 2.$

natural resp, $y_n(t) =$ soln. of $y_h(t)$

$$y_n(t) = 2e^{-t} \longrightarrow ③$$

Forced resp:

For the i/p $x(t) = \cos t$, the particular soln is

$$y_p(t) = K_1 \cos t + K_2 \sin t \longrightarrow ④$$

~~yp(t)...~~

$$\frac{dy_p(t)}{dt} = -K_1 \sin t + K_2 \cos t.$$

$$① \Rightarrow \quad \frac{dy_p(t)}{dt} + y_p(t) = \cos t.$$

$$-K_1 \sin t + K_2 \cos t + K_1 \cos t + K_2 \sin t = \cos t$$

=ing coeff. of $\cos t \Rightarrow K_2 + K_1 = 1$

=ing coeff. of $\sin t \Rightarrow -K_1 + K_2 = 0.$

solving the above eqns: $2K_2 = 1 \Rightarrow K_2 = \frac{1}{2}.$

$\therefore K_1 = \frac{1}{2}.$

$$\therefore \quad y_p(t) = \frac{1}{2}\cos t + \frac{1}{2}\sin t \longrightarrow ⑤$$

Forced resp = homogeneous eqn + particular soln ie $y_f(t) = y_h(t) + y_p(t)$

$$y_f(t) = C_1 e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \longrightarrow ⑥$$

$$y_f(t)\big|_{t=0} = c_1 + \frac{1}{2} = 0 \Rightarrow c_1 = -1/2$$

$$\therefore y_f(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Total response: $y(t) = y_n(t) + y_f(t)$

$$= 2e^{-t} + -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$= \frac{3}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Another method: (For 1st order diff. eqn).

$$\frac{dy(t)}{dt} + y(t) = \cos t \qquad \Big| \qquad \frac{dy}{dt} + P\,y(t) = Q.$$

where $P = 1$ and $Q = \cos t$

Integration factor $IF = e^{\int P\,dt} = e^{t}$

Solution: $y(t) \cdot \cancel{I}F = \int Q \cdot IF \, dt + C.$

$$y(t)\, e^{t} = \int \cos t \cdot e^{t} dt + C.$$

$$y(t)\, e^{t} = \int e^{t} \cdot \cos t \, dt + C.$$

$$\int e^{t} \cos t = \frac{e^{t}}{2}[\sin t + \cos t] \qquad \Big| \qquad \int e^{at}\cos bt\, dt = \frac{e^{at}}{\;}$$

$$\therefore y(t)\, e^{t} = \frac{e^{t}}{2}[\sin t + \cos t] + C.$$

$$y(t) = \frac{1}{2}\sin t + \frac{1}{2}\cos t + e^{-t} \cdot C.$$

$$y(0) = \frac{1}{2} + C = 2 \Rightarrow C = 2 - \frac{1}{2} = 3/2.$$

$$\therefore y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

MODULE 2 & 3

Periodic Signal representation by Fourier Series:
- Continuous time Fourier Series (CTFS).

A continuous time signal $x(t)$ is said to be periodic if there is a positive non zero value of T for which

$$x(t+T) = x(t) \quad \text{for all } t.$$

where T is called fundamental period and $\omega_0 = \frac{2\pi}{T}$ is called fundamental radian frequency.

* Non periodic signals cannot be represented by Fourier series but can be represented by Fourier transform.

Different forms of Fourier series representation:

→ Trigonometric Fourier Series.
→ Complex exponential Fourier Series.

1) Trigonometric Fourier series:

Consider a continuous time signal $x(t)$. This signal can be split up as sines and cosines of fundamental frequency ω_0 and all of its harmonics and expressed as given below:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \quad \longrightarrow (1)$$

Eqn (1) is the Fourier series representation of an arbitrary signal $x(t)$ in trigonometric form.

In eqn (1), a_0 corresponds to the zeroth harmonic or DC value. The expression for the constant term a_0 and the amplitudes of the harmonic can be derived as

$$a_0 = \frac{1}{T} \int_T x(t)\, dt \longrightarrow (2)$$

$$a_K = \frac{2}{T} \int_T x(t) \cos K\omega_0 t\, dt \longrightarrow (3)$$

$$b_K = \frac{2}{T} \int_T x(t) \sin K\omega_0 t\, dt \longrightarrow (4)$$

where $T = \frac{2\pi}{\omega_0}$ is the fundamental period.

$T \in -T/2$ to $T/2$.

To prove the periodicity of $x(t)$:

The periodicity of $x(t)$ is proved if $x(t) = x(t+T)$.

$$(1) \Rightarrow x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin K\omega_0 t.$$

$$x(t+T) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0(t+T) + b_K \sin K\omega_0(t+T)$$

$$= a_0 + \sum_{K=1}^{\infty} a_K \cos(K\omega_0 t + K\omega_0 T) + b_K \sin(K\omega_0 t + K\omega_0 T)$$

W.K.T $T = \frac{2\pi}{\omega_0}$

$$\therefore x(t+T) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(k\omega_0 t + k\omega_0 \frac{2\pi}{\omega_0}\right) + b_k \sin\left(k\omega_0 t + k\omega_0 \frac{2\pi}{\omega_0}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t + 2\pi k) + b_k \sin(k\omega_0 t + 2\pi k)$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t = x(t).$$

Symmetry Conditions:

W.K.T

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \longrightarrow (1)$$

$$a_0 = \frac{1}{T} \int_T x(t)\, dt \longrightarrow (2)$$

$$a_k = \frac{2}{T} \int_T x(t) \cos k\omega_0 t \, dt \longrightarrow (3)$$

$$b_k = \frac{2}{T} \int_T x(t) \sin k\omega_0 t \, dt \longrightarrow (4)$$

Any signal $x(t)$ can be splitted into even and odd functions ie $x(t) = x_e(t) + x_o(t)$

$$\therefore a_0 = \frac{1}{T}\left[\int_{-T/2}^{T/2} x_e(t)\, dt + \int_{-T/2}^{T/2} x_o(t)\, dt\right] \longrightarrow (5)$$

$$a_k = \frac{2}{T}\left[\int_{-T/2}^{T/2} x_e(t) \cos k\omega_0 t\, dt + \int_{-T/2}^{T/2} x_o(t) \cos k\omega_0 t\, dt\right] \longrightarrow (6)$$

$$b_k = \frac{2}{T}\left[\int_{-T/2}^{T/2} x_e(t) \sin k\omega_0 t\, dt + \int_{-T/2}^{T/2} x_o(t) \sin k\omega_0 t\, dt\right] \longrightarrow (7)$$

W.K.T odd function × odd function = even function
even function × even function = even function
even function × odd function = odd function.

For any even function $x_e(t)$, $\int_{-T/2}^{T/2} x_e(t)\,dt = 2\int_0^{T/2} x_e(t)\,dt \rightarrow (8)$

For any odd function, $x_o(t)$, $\int_{-T/2}^{T/2} x_o(t)\,dt = 0 \rightarrow (9)$

If $x(t)$ is an even function, then $x_o(t) = 0$ ie $x(t) = x_e(t)$

$\therefore (5) \Rightarrow a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x_e(t)\,dt = \frac{2}{T}\int_0^{T/2} x(t)\,dt \rightarrow (10)$

$(6) \Rightarrow a_k = \frac{2}{T}\int_{-T/2}^{T/2} x_e(t)\cos k\omega_0 t\,dt$

$a_k = \frac{4}{T}\int_0^{T/2} x(t)\cos k\omega_0 t\,dt \rightarrow (11)$

$(7) \Rightarrow b_k = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{x_e(t)}_{\text{even}}\ \underbrace{\sin k\omega_0 t}_{\text{odd}}\,dt$

W.K.T even × odd = odd & $\int_{-T/2}^{T/2}$ odd function $dt = 0$

$\therefore b_k = 0 \rightarrow (12)$

If $x(t)$ is an odd function, then $x_e(t) = 0$, ie $x(t) = x_o(t)$

$\therefore (5) \Rightarrow a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x_o(t)\,dt = 0$ (from (9)) $\rightarrow (13)$

$(6) \Rightarrow a_k = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{x_o(t)}_{\text{odd}}\ \underbrace{\cos k\omega_0 t}_{\text{even}}\,dt$

W.K.T odd × even = odd & $\int_{-T/2}^{T/2}$ odd function $dt = 0$

$\therefore a_k = 0 \rightarrow (14)$

$(7) \Rightarrow b_k = \frac{2}{T}\int_{-T/2}^{T/2} x_o(t)\sin k\omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} x(t)\sin k\omega_0 t\,dt \rightarrow (15)$

Conclusion:

Thus the Fourier series expansion of an even periodic function contains only cosine terms and a constant → even symmetry. and the fourier series expansion of an odd periodic functions contains only sine terms. → odd symmetry.

Half wave symmetry: A periodic signal which satisfy the condition $x(t) = -x(t \pm T/2)$ is said to have a half wave symmetry.

Complex Exponential Fourier Series:

By using Euler's identity ($e^{j\theta} = \cos\theta + j\sin\theta$), the complex sinusoids can always be expressed in terms of exponentials. DIRS

ie $x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\omega_0 t}$ is called synthesis eqn. where $X(K)$ is called complex fourier coefficient and is expressed as $X(K) = \frac{1}{T}\int_T x(t)\, e^{-jK\omega_0 t}\, dt$ is called analysis eqn.

A periodic waveform $x(t)$ and its fourier coefficient $X(K)$ can be symbolically represented as $x(t) \xleftrightarrow{FS} X(K)$.

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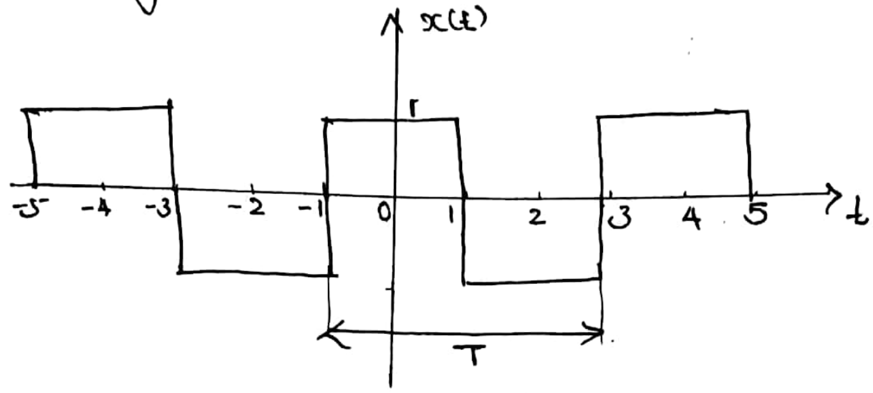
Qn. Find the Trigonometric Fourier series for the periodic signal $x(t)$ shown below.



$T = 4$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

From the figure, $x(t) = x(-t)$ which shows that the given signal is even.

$\therefore \quad b_K = 0$.

W.K.T $\quad x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin \omega_0 t$

$$= a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t.$$

From the figure:

$x(t) = 1 \quad$ for $-1 \le t \le 1$

$\quad = -1 \quad$ for $1 \le t \le 3$

$$a_0 = \frac{1}{T} \int_T x(t)\, dt$$

$$= \frac{1}{4} \int_{-1}^{3} x(t)\, dt$$

$$= \frac{1}{4}\left[\int_{-1}^{1} (1)\, dt + \int_{1}^{3} (-1)\, dt\right]$$

$$= \frac{1}{4}\left[t\right]_{-1}^{1} + \left[-t\right]_{1}^{3} = \frac{1}{4}\left[1 - -(1) + (-3) - (-1)\right]$$

$$= \frac{1}{4}\left[2 + -2\right] = 0$$

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$$a_k = \frac{2}{T}\int_T x(t)\cos k\omega_0 t \, dt$$

$\sin \frac{k\pi}{\cancel{\omega}} = 0$

$$= \frac{2}{4}\int_{-1}^{3} x(t)\cos k\frac{\pi}{2}t \, dt$$

$$= \frac{1}{2}\left[\int_{-1}^{1}\cos k\frac{\pi}{2}t \, dt + \int_{1}^{3}(-1)\cos\frac{k\pi t}{2}\, dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{2}{k\pi}\sin k\frac{\pi}{2}t\right]_{-1}^{1} - \left[\frac{2}{k\pi}\sin\frac{k\pi t}{2}\right]_{1}^{3}\right\}.$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{k\pi}\left[\sin\frac{k\pi}{2} + \sin\frac{k\pi}{2} + \sin\frac{k\pi}{2} + \sin\frac{k\pi}{2}\right]$$

$$= \frac{4}{k\pi}\sin k\pi/2.$$

- $\sin 3k\pi/2$
$= -\sin k\pi/2.$

(OR)

a_k can also be found out by using Symmetry condition. Since $x(t)$ is even,

$$a_k = \frac{4}{T}\int_0^{T/2} x(t)\cos k\omega_0 t \, dt$$

$$= \frac{4}{4}\int_0^{4/2} x(t)\cos k\,\pi/2\,t \, dt \quad \left(\because \begin{array}{l} T=4 \\ \omega_0 = \pi/2 \end{array}\right)$$

$$= \int_0^2 x(t)\cos k\frac{\pi}{2}t\, dt = \int_0^1 \cos k\frac{\pi}{2}t\, dt + \int_1^2 (-1)\cos k\frac{\pi}{2}t\, dt$$

$$= \left[\frac{2}{k\pi}\sin k\frac{\pi}{2}t\right]_0^1 - \left[\frac{2}{k\pi}\sin\frac{k\pi}{2}t\right]_1^2$$

$$= \frac{2}{K\pi}\left[\sin\frac{K\pi}{2} - \sin 0 - \sin \cancel{2}\frac{K\pi}{\cancel{2}} + \sin\frac{K\pi}{2}\right]$$

$$= \frac{2}{K\pi}\left[2\sin\frac{K\pi}{2}\right]$$

$\because \sin 0 = 0$

$\sin K\pi = 0$

$$= \frac{4}{K\pi}\sin\frac{K\pi}{2}.$$

$$\therefore x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\underbrace{\omega_0}_{=\pi/2} t$$

$$= 0 + \sum_{K=1}^{\infty} \frac{4}{K\pi}\sin\frac{K\pi}{2}\cos K\frac{\pi}{2}t$$

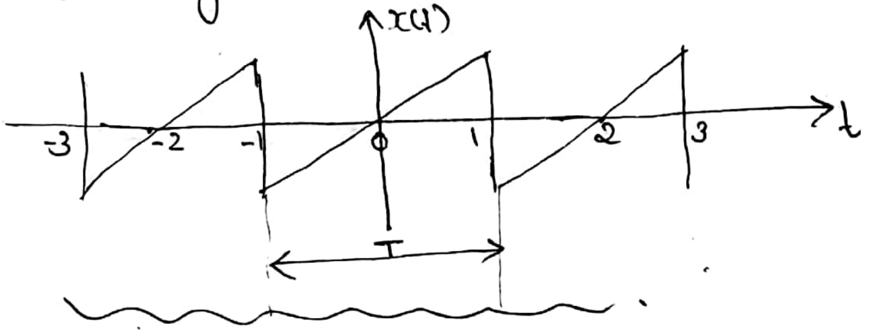
$$= \frac{4}{\pi}\left[\sum_{K=1}^{\infty} \frac{1}{K}\sin\frac{K\pi}{2}\cos K\frac{\pi}{2}t\right].$$

$$= \frac{4}{\pi}\left[\cos\frac{\pi}{2}t + 0 + \frac{1}{3}\left(-\cos\frac{3\pi}{2}t\right) + 0 + \frac{1}{5}\cos\frac{5\pi}{2}t \cdots\right]$$

$$\therefore x(t) = \frac{4}{\pi}\left[\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t + \cdots\right]$$

⑤

Qn: Find the trigonometric Fourier series for the periodic signal $x(t)$ shown below.



$T = 2$, $\omega_0 = \frac{2\pi}{T} = \pi$

from the figure, $x(t) = -x(-t)$ ∴ the signal is an odd signal.

$\therefore \; a_0 = 0 \qquad a_K = 0.$

$$\therefore \; x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin K\omega_0 t.$$

$$x(t) = \sum_{K=1}^{\infty} b_K \sin K\omega_0 t.$$

∵ $x(t)$ is an odd signal,

$$b_K = \frac{4}{T}\int_0^{T/2} x(t) \sin K\omega_0 t \, dt$$

$$= \frac{4}{2}\int_0^{2/2} x(t) \sin K\pi t \, dt$$

$$= 2\int_0^1 x(t) \sin K\pi t \, dt$$

$\int$ of product of 2 functions $= 1^{st} \times \int$ of $2^{nd} - \int\{\text{derivative of } 1^{st} \times \int \text{of } 2^{nd}\} dt$.

$$= 2\int_0^1 t \sin K\pi t \, dt.$$

$$= 2\left[t\left(-\frac{\cos K\pi t}{K\pi}\right) - \int 1\left(-\frac{\cos K\pi t}{K\pi}\right) dt \right]_0^1$$

$$= 2\left[\frac{-t}{K\pi}\cos K\pi t + \frac{1}{K\pi}\frac{\sin K\pi t}{K\pi}\right]_0^1$$

$$= 2\left[-\frac{1}{K\pi}\cos K\pi + \frac{1}{K^2\pi^2}\sin K\pi - (0+0)\right]$$

$$= -\frac{2}{K\pi}\cos K\pi \qquad \because \sin K\pi = 0, \ \sin 0 = 0$$

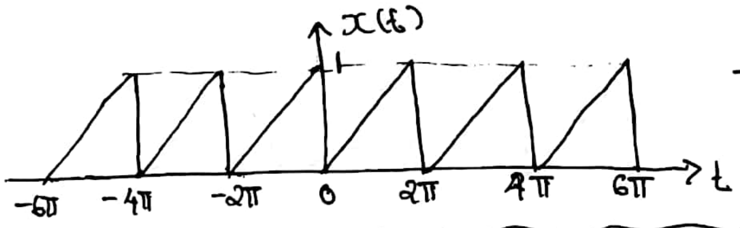
$$x(t) = \sum_{K=1}^{\infty} b_K \sin K\omega_0 t$$

$$= \sum_{K=1}^{\infty} \frac{-2}{K\pi}\cos K\pi \sin K\pi t \qquad \because \omega_0 = \pi$$

$$= \frac{-2}{\pi}\left[-\sin \pi t + \frac{1}{2}\sin 2\pi t + \frac{1}{3}(-\sin 3\pi t) + \cdots\right]$$

$$= \frac{2}{\pi}\left[\sin \pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t + \cdots\right]$$

Qn. Find the trigonometric FS for the periodic signal $x(t)$ shown below.



→ Sawtooth signal.

$T = 2\pi$ and $\omega_0 = \frac{2\pi}{T} = 1$.

From the figure, the signal is neither odd nor even; so the coefficients a_0, a_K and b_K are to be evaluated.

For a ramp signal the slope is $\frac{1}{2\pi}$

$\therefore x(t) = \frac{t}{2\pi}$

$$a_0 = \frac{1}{T}\int_T x(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\, dt = \frac{1}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{8\pi^2}\left[4\pi^2 - 0\right] = \frac{1}{8\pi^2}4\pi^2 = \frac{1}{2}.$$

$$a_k = \frac{2}{T}\int_T x(t)\cos k\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos kt\, dt \qquad \Big|\ \because\ \omega_0 = 1$$

$$= \frac{1}{2\pi^2}\left[t\left(\frac{\sin kt}{k}\right) - \int 1 \cdot \frac{\sin kt}{k}\, dt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{t}{k}\sin kt - \frac{1}{k}\left(-\frac{\cos kt}{k}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{t}{k}\sin kt + \frac{1}{k^2}\cos kt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[\frac{2\pi}{k}\underbrace{\sin 2\pi k}_{=0} + \frac{1}{k^2}\underbrace{\cos 2\pi k}_{=1} - \left(0 + \frac{1}{k^2}\underbrace{\cos 0}_{=1}\right)\right]$$

$$= \frac{1}{2\pi^2}\left[0 + \frac{1}{k^2} - 1/k^2\right] = \underline{\underline{0}}.$$

$$b_k = \frac{2}{T}\int_T x(t)\sin k\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin kt\, dt = \frac{1}{2\pi^2}\left[t\left(-\frac{\cos kt}{k}\right) - \int 1\left(-\frac{\cos kt}{k}\right)dt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[-\frac{t}{k}\cos kt + \frac{1}{k^2}\sin kt\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[-\frac{2\pi}{K}\underbrace{\cos 2\pi k}_{=1} + \frac{1}{K^2}\underbrace{\sin 2\pi k}_{=0} - \left(0+\frac{1}{K^2}\underbrace{\sin 2\pi i}_{=0}\right.\right.$$

$$= \frac{1}{2\pi^2}\left(-\frac{2\pi}{K}\right) = -\frac{1}{K\pi}$$

$$\therefore x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t + b_K \sin K\omega_0 t$$

$$= \frac{1}{2} + \sum_{K=1}^{\infty} 0 - \frac{1}{K\pi}\sin Kt$$

$\because a_0 = 1/2$
$a_K = 0$
$\omega_0 = 1$

$$\underbrace{x(t) = \frac{1}{2} - \sum_{K=1}^{\infty}\frac{1}{K\pi}\sin Kt}$$

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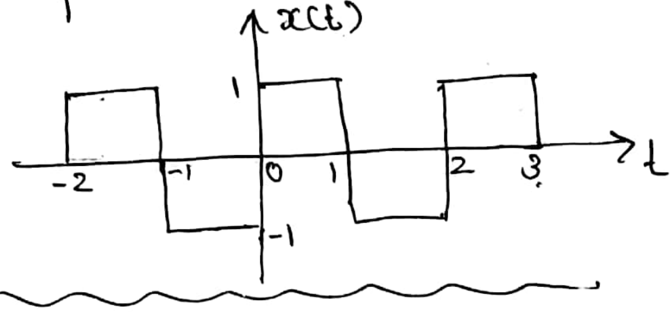
Qn. Obtain the exponential fourier series representation for the waveform $x(t)$ shown in figure.



from the figure $T = 2$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$X(K) = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2}\int_0^2 x(t)\, e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\left[\int_0^1 e^{-jk\pi t}\, dt + \int_1^2 -e^{-jk\pi t}\, dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 - \left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_1^2\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-jk\pi}}{-jk\pi} - \frac{e^{0}}{-jk\pi}\right] - \left[\frac{e^{-jk2\pi}}{-jk\pi} - \frac{e^{-jk\pi}}{-jk\pi}\right]\right\}$$

$$= \frac{1}{-2jk\pi}\left[e^{-jk\pi} - 1 - \underbrace{e^{-jk2\pi}}_{=1} + e^{-jk\pi}\right]$$

$$= \frac{-1}{2jk\pi}\left[2e^{-jk\pi} - 1 - 1\right] = \frac{-1}{2jk\pi}\left[2e^{-jk\pi} - 2\right]$$

$$= \frac{1}{jk\pi}\left[1 - e^{-jk\pi}\right] = \frac{1}{jk\pi}\left[1 - (\cos k\pi - j\sin k\pi)\right]$$

$$= \frac{1}{jk\pi}\left[1 - \cos k\pi\right] \quad = 0, \text{ if } k \text{ is even.}$$

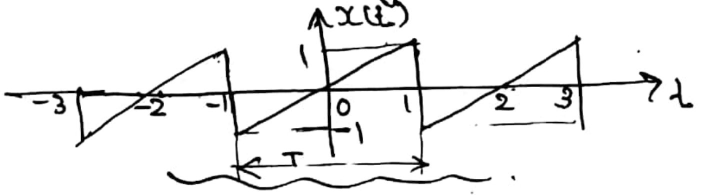
$$= \frac{2}{jk\pi}, \text{ if } k \text{ is odd.}$$

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\pi t}$$

$$= \sum_{K=-\infty}^{\infty} \frac{2}{jK\pi} e^{jK\pi t} = \underbrace{\frac{2}{j\pi} \sum_{K=-\infty}^{\infty} \frac{1}{K} e^{jK\pi t}}$$

||5.6||

Qn. Find the complex fourier coefficient for the signal $x(t)$ shown in fig.



from the figure, $T = 2$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

$x(t) = t, \quad -1 < t < 1$

$$X(K) = \frac{1}{T} \int_T x(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{2} \int_{-1}^{1} x(t)\, e^{-jK\pi t}\, dt = \frac{1}{2} \int_{-1}^{1} t\, e^{-jK\pi t}\, dt$$

$$= \frac{1}{2} \int_{-1}^{1} t \left[\cos K\pi t - j \sin K\pi t\right] dt.$$

The waveform $x(t)$ is odd and hence $X(K)$ is purely imaginary ie $X(K) = jB(K)$. $\{A(K) = 0$

$$\therefore X(K) = \frac{2}{2} \int_0^1 -jt \sin K\pi t\, dt = \int_0^1 -jt \sin K\pi t\, dt$$

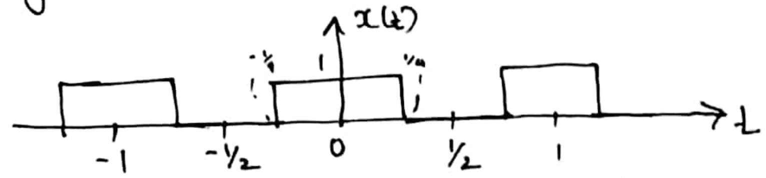
$$= -j \left[t \cdot \left(\frac{-\cos K\pi t}{K\pi}\right) - \int 1 \cdot \left(\frac{-\cos K\pi t}{K\pi}\right) dt \right]_0^1$$

$$= -j \left[\frac{-t}{K\pi} \cos K\pi t + \frac{\sin K\pi t}{K^2\pi^2} \right]_0^1$$

$$= -j \left[\frac{-1}{K\pi} \cos K\pi + \underbrace{\frac{\sin K\pi}{K^2\pi^2}}_{=0} - \underbrace{\left(0 - \frac{\sin 0}{K^2\pi^2}\right)}_{=0} \right]$$

$$= -j \left[\frac{-1}{K\pi} \cos K\pi \right] = \frac{j}{K\pi} \cos K\pi$$

Find the Fourier series representation for the signal $x(t)$:



$T = 1 \qquad \omega = \frac{2\pi}{T} = 2\pi$

$$X(K) = \frac{1}{T}\int_T x(t)\, e^{-jK\omega t}\, dt$$

$$= \int_{-1/4}^{1/4} 1 \cdot e^{-jK\omega t}\, dt = \left[\frac{e^{-jK\omega t}}{-jK\omega}\right]_{-1/4}^{1/4}$$

$$= \frac{1}{-jK\omega}\left[e^{-jK\frac{\omega}{4}} - e^{jK\frac{\omega}{4}}\right]$$

$$= \frac{1}{jK\omega}\left[e^{jK\frac{\omega}{4}} - e^{-jK\frac{\omega}{4}}\right]$$

$$= \frac{1}{\cancel{j}K\omega} \cdot 2\cancel{j} \sin K\frac{\omega}{4} \qquad (\because \omega = 2\pi)$$

$$= \frac{2}{K \cdot \frac{2\pi}{\cancel{T}}} \cdot \sin K \cdot \frac{2\pi}{4} = \frac{\sin K\frac{\pi}{2}}{K\pi}$$

$$\cancel{= \frac{1}{K\pi}}$$

$$\sum_{K=-\infty}^{\infty} (-1)^{2K+1}$$

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\omega t}$$

$$= \sum_{K=-\infty}^{\infty} \frac{\sin K\pi/2}{K\pi}\, e^{jK \cdot 2\pi t}$$

$$\frac{e^{jK\pi/2} - e^{-jK\pi/2}}{K}$$

$$= \frac{1}{\pi}\left[-\frac{e^{-j2\pi t}}{-1} + \frac{e^{j2\pi t}}{1}\right]$$

$$= \frac{1}{\pi}\left[e^{j2\pi t} + e^{-j2\pi t}\right]$$

$$= \frac{2}{\pi} \cos 2\pi t$$

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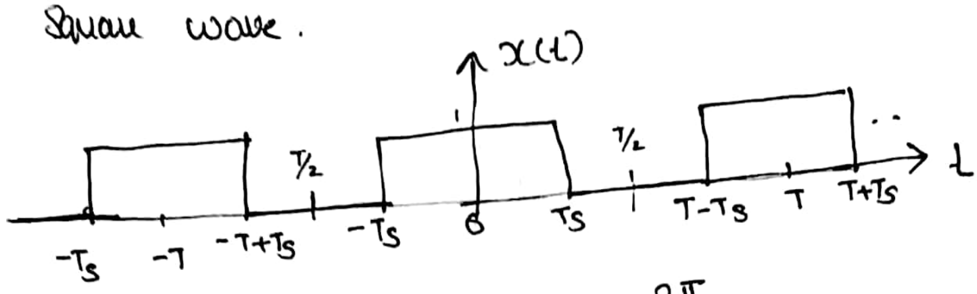
Qn. Determine the Fourier Series representation of a square wave.



period $= T$, $\omega = \frac{2\pi}{T}$

$$X(K) = \frac{1}{T}\int_T x(t)\, e^{-jK\omega t}\, dt$$

$$= \frac{1}{T}\int_{-Ts}^{Ts} x(t)\, e^{-jK\omega t}\, dt$$

$$= \frac{1}{T}\int_{-Ts}^{Ts} 1 \cdot e^{-jK\omega t}\, dt = \frac{1}{T}\left[\frac{e^{-jK\omega t}}{-jK\omega}\right]_{-Ts}^{Ts}$$

$$= \frac{1}{-TjK\omega}\left[e^{-jK\omega Ts} - e^{jK\omega Ts}\right]$$

$$= \frac{1}{TjK\omega}\left[e^{jK\omega Ts} - e^{-jK\omega Ts}\right] \quad (\because \omega = \tfrac{2\pi}{T})$$

$$= \frac{1}{T\cancel{j}K\omega}\, 2\cancel{j}\sin K\omega Ts = \frac{2\sin K\omega Ts}{TK\omega} = \frac{2\sin K\frac{2\pi}{T}Ts}{\cancel{T}K\frac{2\pi}{\cancel{T}}}$$

$$= \frac{\cancel{2}\sin K\frac{2\pi}{T}Ts}{\cancel{2}\pi K}$$

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\omega t}$$

$$= \sum_{K=-\infty}^{\infty} \frac{\cancel{2}\sin K\frac{2\pi}{T}Ts}{\cancel{2}\pi K}\, e^{jK\omega t} = \frac{1}{\pi}\sum_{K=-\infty}^{\infty} \frac{\sin K\frac{2\pi}{T}Ts}{\pi K}\, e^{jK\omega t}$$

Amplitude and phase Spectra of a periodic Signal:

$X(K) = A(K) + jB(K)$

magnitude, $|X(K)| = \sqrt{A^2(K) + B^2(K)}$

Phase, $\angle X(K) = \tan^{-1}\left(\frac{B(K)}{A(K)}\right)$

A plot of $|X(K)|$ versus K is called amplitude Spectrum and a plot of $\angle X(K)$ versus K is called phase Spectrum of periodic Signal.

Properties of Fourier Series:

1) linearity:

If $x(t) \xleftrightarrow{FS} X(K)$ and $y(t) \xleftrightarrow{FS} Y(K)$

then $z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{FS} Z(K) = a\,X(K) + b\,Y(K)$.

Proof: $Z(K) = \frac{1}{T}\int_T z(t)\, e^{-jK\omega_0 t}\, dt$

$$= \frac{1}{T}\int_T [a\,x(t) + b\,y(t)]\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T a\,x(t)\, e^{-jK\omega_0 t}\, dt + \frac{1}{T}\int_T b\,y(t)\, e^{-jK\omega_0 t}\, dt$$

$$= a\underbrace{\frac{1}{T}\int_T x(t)\, e^{-jK\omega_0 t}\, dt}_{X(K)} + b\underbrace{\frac{1}{T}\int_T y(t)\, e^{-jK\omega_0 t}\, dt}_{Y(K)}$$

$$= a\,X(K) + b\,Y(K).$$

2) Time Shift:

If $x(t) \xleftrightarrow{FS} X(K)$, then $z(t) = x(t - t_0) \xleftrightarrow{FS} Z(K) = e^{-jK\omega_0 t_0} X(K)$

Proof: $Z(K) = \frac{1}{T}\int_T z(t)\, e^{-jK\omega_0 t}\, dt$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

Put $\lambda = t - t_0$

$d\lambda = dt$

$$\therefore Z(K) = \frac{1}{T}\int_T x(\lambda)\, e^{-jk\omega_0(\lambda + t_0)}\, d\lambda$$

$$= \frac{1}{T}\, e^{-jk\omega_0 t_0}\int_T x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda.$$

$$= e^{-jk\omega_0 t_0}\, \underbrace{\frac{1}{T}\int_T x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda}_{X(K)}$$

$$= e^{-jk\omega_0 t_0}\, X(K).$$

3) frequency Shift:

If $x(t) \xleftrightarrow{FS} X(K)$, then $z(t) = e^{jK_0\omega_0 t}\, x(t) \xleftrightarrow{FS}$ $Z(K) = X(K-K_0)$.

Proof: $$Z(K) = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T e^{jK_0\omega_0 t}\, x(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-j(K-K_0)\omega_0 t}\, dt$$

$$= X(K-K_0).$$

4) Scaling:

If $x(t) \xleftrightarrow{FS} X(K)$

then $z(t) = x(at) \xleftrightarrow{FS} Z(K) = X(K)$.

5) Convolution:

If $x(t) \xleftrightarrow{FS} X(K)$ and $y(t) \xleftrightarrow{FS} Y(K)$

Then $z(t) = x(t) \circledast y(t) \xleftrightarrow{FS} Z(K) = T\,X(K)\,Y(K)$.

Proof: $Z(K) = \frac{1}{T}\int_{t=T} z(t)\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{T}\int_{t=T} x(t) \circledast y(t)\, e^{-jk\omega_0 t}\, dt$

W.K.T $x(t) \circledast y(t) = \int_{l=T} x(l)\, y(t-l)\, dl$

$\therefore Z(K) = \frac{1}{T}\int_{t=T}\left[\int_{l=T} x(l)\, y(t-l)\, dl\right] e^{-jk\omega_0 t}\, dt$

changing the order of integration

$Z(K) = \frac{1}{T}\int_{l=T} x(l) \int_{t=T} y(t-l)\, e^{-jk\omega_0 t}\, dt\, dl$

Put $m = t-l$

$\frac{dm}{dt} = 1 \Rightarrow dm = dt$

$\therefore Z(K) = \frac{1}{T}\int_{l=T} x(l) \int_{m=T} y(m)\, e^{-jk\omega_0 (m+l)}\, dm\, dl$

$= \underbrace{\frac{1}{T}\int_{l=T} x(l)\, e^{-jk\omega_0 l}\, dl}_{X(K)}\ \underbrace{\int_{m=T} y(m)\, e^{-jk\omega_0 m}\, dm}_{T\,Y(K)}$

$= X(K)\; T\, Y(K)$

6) multiplication or modulation

If $x(t) \xleftrightarrow{FS} X(K)$ and $y(t) \xleftrightarrow{FS} Y(K)$ then

$z(t) = x(t).y(t) \xleftrightarrow{FS} Z(K) = X(K) * Y(K)$.

Proof: $Z(K) = \frac{1}{T}\int_{T} z(t)\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{T}\int_{T} x(t).y(t)\, e^{-jk\omega_0 t}\, dt$.

We have the synthesis eqn: $x(t) = \sum_{l=-\infty}^{\infty} X(l)\, e^{jl\omega_0 t}$

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$$\therefore Z(K) = \frac{1}{T}\int_T \left[\sum_{l=-\infty}^{\infty} X(l) e^{jl\omega_0 t}\right] y(t) e^{-jk\omega_0 t} dt$$

changing the order of summation and integration

$$Z(K) = \frac{1}{T} \sum_{l=-\infty}^{\infty} X(l) \int_T y(t) e^{-j(K-l)\omega_0 t} dt$$

$$= \sum_{l=-\infty}^{\infty} X(l) Y(K-l) = \underbrace{X(K) * Y(K)}.$$

7) Parseval's theorem:

If $x(t) \xleftrightarrow{FS} X(K)$, then the average power,

$$P = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{K=-\infty}^{\infty} |X(K)|^2$$

Proof: $P = \frac{1}{T}\int_T |x(t)|^2 dt$

$$= \frac{1}{T}\int_T x(t) x^*(t) dt$$

We know that $x(t) = \sum_{K=-\infty}^{\infty} X(K) e^{jK\omega_0 t}$

Taking conjugate on both sides

$$x^*(t) = \sum_{K=-\infty}^{\infty} X^*(K) e^{-jK\omega_0 t}$$

$$\therefore \text{average power, } P = \frac{1}{T}\int_T x(t) \left[\sum_{K=-\infty}^{\infty} X^*(K) e^{-jK\omega_0 t}\right] dt$$

changing order of summation and integration

$$P = \sum_{K=-\infty}^{\infty} X^*(K) \underbrace{\frac{1}{T}\int_T x(t) e^{-jK\omega_0 t} dt}_{X(K)}$$

$$= \sum_{K=-\infty}^{\infty} X^*(K) X(K)$$

$$= \underbrace{\sum_{K=-\infty}^{\infty} |X(K)|^2}$$

* Power Spectral density:

A plot of $|X(K)|^2$ versus K is known as power spectral density.

Fourier representation for non periodic signals:
- Continuous time fourier transform (CTFT)

The CTFT of a non periodic signal $x(t)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \longrightarrow \text{Analysis eqn.}$$

The inverse CTFT of $X(\omega)$ is given by.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \longrightarrow \text{Synthesis eqn.}$$

Amplitude and phase Spectra:

A plot of $|X(\omega)|$ versus ω is called magnitude spectrum and a plot of $\angle X(\omega)$ versus ω is called phase spectrum.

Qn. Find the Fourier transform of the signal $x(t) = \delta(t)$. Also plot magnitude and phase spectra.

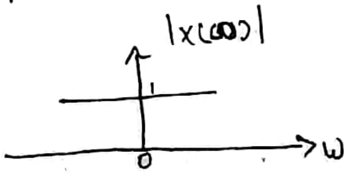
Given $x(t) = \delta(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = 1$$

$|X(\omega)| = 1$

$\angle X(\omega) = 0$



$\Rightarrow$ magnitude spectrum.

Q. Find the Fourier transform of the signal $x(t) = \delta(t+0.5) - \delta(t-0.5)$. Also plot the magnitude and phase spectra.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [\delta(t+0.5) - \delta(t-0.5)]\, e^{-j\omega t}\, dt$$

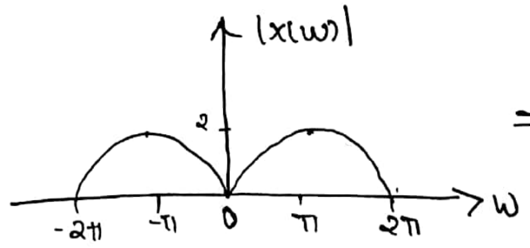
$$= e^{-j\omega t}\Big|_{t=-0.5} - e^{-j\omega t}\Big|_{t=0.5}$$

$$= e^{+j0.5W} - e^{-j0.5W} = 2j\sin(0.5W)$$

$$|X(\omega)| = \sqrt{0^2 + 4\sin^2(0.5W)} = 2\sin 0.5W.$$

$$\angle X(\omega) = \tan^{-1}\left(\frac{2\sin(0.5W)}{0}\right)$$

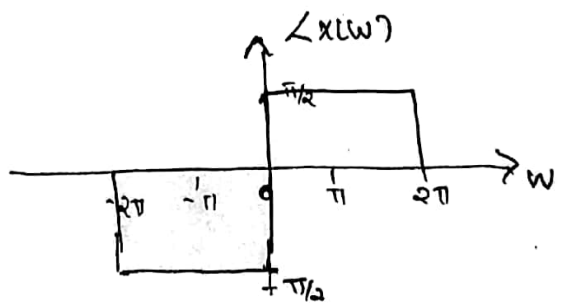
ω	-2π	$-\pi$	0	π	2π
$\lvert X(\omega)\rvert$	0	2	0	2	0
$\angle X(\omega)$	$-\pi/2$	$-\pi/2$	$\pm\pi/2$	$\pi/2$	$\pi/2$



⇒ magnitude Spectrum

$\sqrt{(2\sin 0.5(\pi))^2}$

$\sqrt{4(-\sin\pi)^2}$



Properties of CTFT:

1) linearity:

If $x(t) \xleftrightarrow{FT} X(\omega)$

$y(t) \xleftrightarrow{FT} Y(\omega)$

then $z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{FT} Z(\omega) = a\,X(\omega) + b\,Y(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [a\,x(t) + b\,y(t)]\, e^{-j\omega t}\, dt$$

$$= a \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}_{X(\omega)} + b \underbrace{\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt}_{Y(\omega)}$$

$$Z(\omega) = a\,X(\omega) + b\,Y(\omega)$$

2) Time Shift:

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $z(t) = x(t-t_0) \xleftrightarrow{FT} Z(\omega) = e^{-j\omega t_0} X(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$Z(\omega) = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

Put $\lambda = t - t_0 \Rightarrow t = \lambda + t_0$

$\frac{dt}{d\lambda} = 1 \Rightarrow dt = d\lambda$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega(\lambda + t_0)}\, d\lambda = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, e^{-j\omega t_0}\, d\lambda$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda = e^{-j\omega t_0} X(\omega)$$

3) Frequency shift:

If $x(t) \xleftrightarrow{FT} X(\omega)$, then $z(t) = e^{j\omega_0 t} x(t) \xleftrightarrow{FT} Z(\omega) = X(\omega-\omega_0)$

Proof: $\cancel{z(t) = e^{j\omega_0 t} x(t) \xleftrightarrow{FT}}$

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(\omega-\omega_0)$$

4) Convolution:

If $x(t) \xleftrightarrow{FT} X(\omega)$

$y(t) \xleftrightarrow{FT} Y(\omega)$

then $z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(\omega) = X(\omega) Y(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)] e^{-j\omega t} dt$$

W.K.T $x(t) * y(t) = \int_{-\infty}^{\infty} x(l) y(t-l) dl$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(l) y(t-l) dl \right] e^{-j\omega t} dt$$

Interchanging the order of integration;

$$Z(\omega) = \int_{-\infty}^{\infty} x(l) \int_{-\infty}^{\infty} y(t-l) e^{-j\omega t} dt \, dl$$

Put $m = t-l \Rightarrow t = m+l$; $dm = dt$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} x(l) \int_{-\infty}^{\infty} y(m) e^{-j\omega(m+l)} dm \, dl$$

$$= \int_{-\infty}^{\infty} x(l) \int_{-\infty}^{\infty} y(m) e^{-j\omega m} \cdot e^{-j\omega l} dm \, dl$$

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$$Z(w) = \underbrace{\int_{-\infty}^{\infty} x(l)\, e^{-jwl}\, dl}_{X(w)} \cdot \underbrace{\int_{-\infty}^{\infty} y(m)\, e^{-jwm}\, dm}_{Y(w)}$$

$$Z(w) = X(w) \cdot Y(w)$$

5) Multiplication:

$$\text{If } x(t) \xleftrightarrow{FT} X(w)$$

$$y(t) \xleftrightarrow{FT} Y(w)$$

then $z(t) = x(t) \cdot y(t) \xleftrightarrow{FT} Z(w) = \frac{1}{2\pi}\left[X(w) * Y(w)\right]$

Proof:

$$Z(w) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t) \cdot y(t)]\, e^{-j\omega t}\, dt.$$

W.K.T $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\, e^{j\omega_0 t}\, d\omega_0$

$$\therefore Z(w) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\, e^{j\omega_0 t}\, d\omega_0\right] y(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration

$$Z(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\left[\int_{-\infty}^{\infty} y(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt\right] d\omega_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\left[\int_{-\infty}^{\infty} y(t)\, e^{-j(\omega-\omega_0)t}\, dt\right] d\omega_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w_0)\, Y(w-w_0)\, d w_0$$

$$Z(w) = \frac{1}{2\pi}\left[X(w) * Y(w-w_0)\right]$$

6. Frequency differentiation:

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $\underbrace{-jt\, x(t)}_{z(t)=} \xleftrightarrow{FT} \underbrace{\frac{d}{d\omega} X(\omega)}_{Z(\omega)}$.

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} -jt\, x(t)\, e^{-j\omega t}\, dt$$

$$= \frac{d}{d\omega} X(\omega).$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} -jt\, x(t)\, e^{-j\omega t}\, dt$$

6. Parseval's theorem:

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Proof:

LHS: $\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$

W.K.T $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$.

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega$$

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega\right] dt$$

Interchanging the order of integration

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}_{X(\omega)}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \text{RHS}.$$

Q. What is the energy of the signal $x(t) = e^{-at} u(t)$, also find the fourier transform.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |e^{-at} u(t)|^2 dt$$

$$= \int_{0}^{\infty} e^{-2at} dt = \frac{-1}{2a} \left[e^{-2at} \right]_0^{\infty}$$

$$= \frac{-1}{2a} [0-1] = \frac{1}{2a}.$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-t[a+j\omega]} dt$$

$$= \left[\frac{e^{-t[a+j\omega]}}{a+j\omega} \right]_0^{\infty}$$

$$= \frac{-1}{a+j\omega} [0-1] = \underline{\underline{\frac{1}{a+j\omega}}}.$$

* **Energy spectral density:**

A plot of $|X(\omega)|^2$ versus ω is called energy spectral density.

conjugation and conjugation symmetry property:

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

Proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{-j(-\omega)t}\, dt$$

If $x(t)$ is real $x^*(t) = x(t)$.

$$\therefore X^*(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega)t}\, dt$$

$$= X(-\omega).$$

Also $X^*(-\omega) = X(\omega)$

If Show that FT of a conjugate symmetric signal is purely real.

Existence of Fourier Integral.

Existence of Fourier Series: (Dirichlet conditions)

The conditions under which a periodic signal can be represented by a Fourier series are known as Dirichlet conditions.

In each period,

1) $x(t)$ has only a finite no. of maxima & minima
2) $x(t)$ has a finite no. of discontinuities.
3) $x(t)$ is absolutely integrable over one period,

ie $\int_0^T |x(t)|\, dt < \infty$.

Existence of Fourier transform:

The Fourier transform does not exist for all aperiodic functions. The conditions for a $x(t)$ to have Fourier transform are

1) $x(t)$ is absolutely integrable over $(-\infty, \infty)$

ie $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

2) $x(t)$ has finite no. of discontinuities.

3) $x(t)$ has a finite no. of maxima and minima.

Fourier transform theorems:

1) Convolution theorems.

a) Time convolution

b) frequency convolution (modulation) (multiplication).

2) Parseval's theorem

(Rayleigh's theorem)

<u>Proof</u>: Refer properties of CTFT.

Frequency response of LTI systems:

The frequency response gives the magnitude response and phase response of the system.

Frequency response (Transfer function), $H(\omega) = \dfrac{Y(\omega)}{X(\Omega)}$.

A plot of $|H(\omega)|$ versus ω is called magnitude spectrum and a plot of $\angle H(\omega)$ versus ω is called phase spectrum.

Qn. Find the frequency response of the system described by the differential eqn: $\dfrac{d^3y(t)}{dt^3} + 6\dfrac{d^2y(t)}{dt^2} + 5\dfrac{dy(t)}{dt} + 4y(t) = 3x(t)$

Taking FT on both sides,

$$\frac{d^3y(t)}{dt^3} \xleftrightarrow{FT} (j\omega)^3 Y(\omega)$$

$$\frac{d^2y(t)}{dt^2} \xleftrightarrow{FT} (j\omega)^2 Y(\omega)$$

$$\frac{dy(t)}{dt} \xleftrightarrow{FT} j\omega Y(\omega)$$

$$(j\omega)^3 Y(\omega) + 6(j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 4Y(\omega) = 3X(\omega).$$

$$Y(\omega)\left[(j\omega)^3 + 6(j\omega)^2 + 5j\omega + 4\right] = 3X(\omega).$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3}{(j\omega)^3 + 6(j\omega)^2 + 5(j\omega) + 4}$$

↓
freq. response.

Qn. The i/p and output of a causal LTI system are described by the differential eqn:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

a) Find the frequency response of the system.

b) Find the impulse response of the system

c) What is the response of the s/m if $x(t) = te^{-t}u(t)$.

Soln: $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$

Taking FT

$(jw)^2 Y(w) + 3(jw)Y(w) + 2Y(w) = X(w)$

$Y(w)\left[(jw)^2 + 3(jw) + 2\right] = X(w).$

$\left.\begin{array}{l} H(w) \\ \text{freq. resp.} \end{array}\right\} = \frac{Y(w)}{X(w)} = \frac{1}{(jw)^2 + 3(jw) + 2} = \frac{1}{(jw+2)(jw+1)}$

b) impulse response h(t).

$H(w) = \frac{1}{(jw+2)(jw+1)} = \frac{A}{(jw+2)} + \frac{B}{jw+1}$

$1 = A(jw+1) + B(jw+2).$

Put $jw = -2 \Rightarrow -A = 1 \Rightarrow A = 1$

Put $jw = -1 \Rightarrow B = 1$

$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{jw+a}.$

$\therefore H(w) = \frac{-1}{jw+2} + \frac{1}{jw+1}$

Taking inverse FT

$h(t) = -e^{-2t}u(t) + e^{-t}u(t)$

c) Given $x(t) = te^{-t}u(t).$

$\therefore X(w) = \frac{1}{(jw+1)^2}.$

freq. diff. property.

$-jtx(t) \xleftrightarrow{FT} \frac{d}{dw}X(w)$

$tx(t) \xleftrightarrow{FT} \frac{1}{-j}\frac{d}{dw}X(w).$

$te^{-t}u(t) \xleftrightarrow{FT} \frac{1}{-j}\frac{d}{dw}\frac{1}{jw+1}$

$= \frac{1}{-j}\frac{(jw+1)\times 0 - 1\cdot j}{(jw+1)^2}$

$= \frac{1}{(jw+1)^2}$

$Y(w) = H(w) \times X(w).$

$\boxed{t^{n}e^{-at} = \frac{n!}{(jw+a)^{n+1}}}$

$= \frac{1}{(jw+2)(jw+1)} \cdot \frac{1}{(jw+1)^2}$

$= \frac{1}{(jw+2)(jw+1)^3} = \frac{A}{jw+2} + \frac{B}{jw+1} + \frac{C}{(jw+1)^2} + \frac{D}{(jw+1)^3}$

$A = -1, \quad B = 1, \quad C = -1, \quad D = 1$

$\therefore Y(w) = -\frac{1}{jw+2} + \frac{1}{jw+1} - \frac{1}{(jw+1)^2} + \frac{1}{(jw+1)^3}.$

Taking inv. FT $\Rightarrow y(t) = -e^{-2t}u(t) + e^{-t}u(t) - te^{-t}u(t) + \frac{t^2}{2}e^{-t}u(t)$

Qn. Consider a causal LTI s/m with frequency response $H(\omega) = \frac{1}{j\omega+3}$. For a particular i/p $x(t)$, the s/m is observed to produce the output $y(t) = e^{-t}u(t) - e^{-2t}u(t)$.

Determine $x(t)$.

Given $y(t) = e^{-t}u(t) - e^{-2t}u(t)$

Taking FT.

$$Y(\omega) = \frac{1}{j\omega+1} - \frac{1}{j\omega+2} = \frac{j\omega+2-j\omega-1}{(j\omega+1)(j\omega+2)}$$

$$= \frac{1}{(j\omega+1)(j\omega+2)}.$$

$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{j\omega+a}$

$t\,e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(j\omega+a)^2}$

$t^2 e^{-at}u(t) \xleftrightarrow{FT}$

W.K.T, $H(\omega) = \frac{Y(\omega)}{X(\omega)} \Rightarrow X(\omega) = \frac{Y(\omega)}{H(\omega)}$.

$$= \frac{j\omega+3}{(j\omega+1)(j\omega+2)} = \frac{A}{j\omega+1} + \frac{B}{j\omega+2}.$$

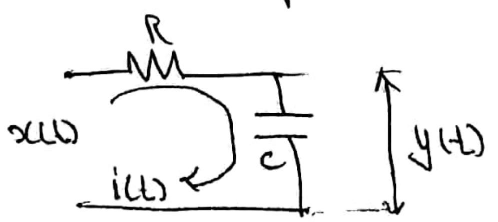
$j\omega+3 = A(j\omega+2) + B(j\omega+1)$

Put $j\omega = -1 \Rightarrow A = 2$, Put $j\omega = -2 \Rightarrow B = -1$

$$\therefore X(\omega) = 2\,\frac{1}{j\omega+1} - \frac{1}{j\omega+2}.$$

Taking inv. LT $\Rightarrow x(t) = 2e^{-t}u(t) - e^{-2t}u(t)$

Qn. Find the frequency response of the RC circuit shown in fig. given below. Plot magnitude and phase response for RC = 1. Also find the impulse response of the circuit.



The differential eqn. governing the response of the circuit is $x(t) = Ri(t) + \frac{1}{c}\int i(t)\,dt$

$$y(t) = \frac{1}{c}\int i(t)\,dt$$

Taking FT on both sides of the above eqns:

$$X(\omega) = R\,I(\omega) + \frac{1}{c}\frac{I(\omega)}{j\omega} \Rightarrow I(\omega)\left[R + \frac{1}{j\omega c}\right] = X(\omega)$$

$$X(\omega) = \left[\frac{j\omega RC+1}{j\omega C}\right] I(\omega)$$

$$Y(\omega) = \frac{1}{c}\frac{I(\omega)}{j\omega}.$$

freq. response, $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{I(\omega)}{j\omega C}\cdot\left[\frac{j\omega C}{j\omega RC+1}\right] I(\omega)$

$$H(\omega) = \frac{1}{1+j\omega RC}.$$

impulse response: $H(\omega) = \frac{1}{RC\left[j\omega + \frac{1}{RC}\right]}$.

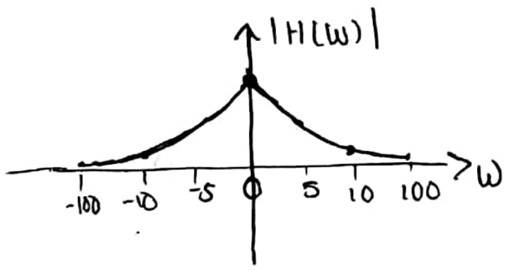
Taking inv. LT $\Rightarrow \underset{\text{(imp. response)}}{h(t)} = \frac{1}{RC}\cdot e^{-\frac{1}{RC}t}\,u(t)$.

when $RC = 1$ $H(\omega) = \frac{1}{1+j\omega}$.

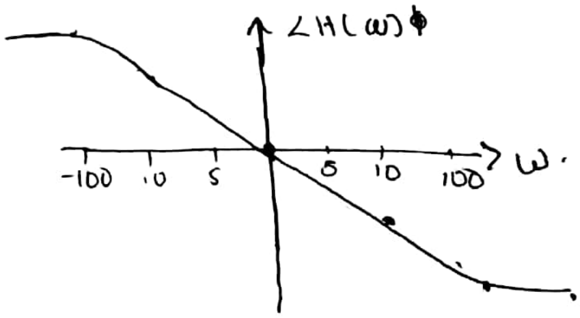
magnitude response $|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$

Phase response $\angle H(\omega) = -\tan^{-1}\omega$.

$\omega \to$	0	5	10	20	50	100	∞
$\lvert H(\omega)\rvert$	1		0.1		0.02	0.01	0
$\angle H(\omega)$	0		-1.47		-1.55	-1.56	-1.57



Magnitude response



Phase response.

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Correlation theory:

Correlation is basically used to compare two signals. It is a measure of the degree to which two signals are similar.

The correlation of two signals is divided into

→ cross correlation

→ Auto-correlation.

Cross correlation:

It is a measure of similarity between one signal and the time delayed version of another signal. The cross correlation of two different signals $x(t)$ and $y(t)$ is given by

$$r_{xy}(l) = \int_{-\infty}^{\infty} x(t)\, y(t-l)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y[-(l-t)]\, dt$$

$$= x(l) * y(-l)$$

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)\, y(\tau-t)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, y(-(t-\tau))\, d\tau$$

$$= x(t) * y(-t).$$

Auto-correlation:

When $x(t) = y(t)$, the correlation operation is called autocorrelation. That is, it is defined as the correlation of a signal with itself. The auto-correlation of ~~two~~ signal $x(t)$ is given by

$$r_{xx}(l) = \int_{-\infty}^{\infty} x(t) \cdot x(t-l)\, dt$$

The time shift $l=0$, then

$$r_{xy}(0) = \int_{-\infty}^{\infty} x(t)\, x(t)\, dt$$

and

$$r_{xx}(0) = \int_{-\infty}^{\infty} x^2(t)\, dt.$$

Correlation theorem:

* The cross correlation of two signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of FT of second signal

$$r_{xy}(t) \xleftrightarrow{FT} X_x(\omega)\, X_y^*(\omega).$$

FT of cross correlation $r_{xy}(t)$ with multiplied

* The autocorrelation theorem states that the FT of autocorrelation function $r_{xx}(t)$ yields the energy density function of signal $x(t)$

$$r_{xx}(t) \xleftrightarrow{F} |X(\omega)|^2$$

Qn. Determine the autocorrelation function of $x(t) = e^{-at}u(t)$

$$|X(\omega)|^2 = X(\omega) \cdot X^*(\omega)$$

$$x(t) = e^{-at}u(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

$$= \frac{1}{a+j\omega}.$$

$$|X(\omega)|^2 = \frac{1}{a+j\omega} \cdot \frac{1}{a-j\omega}$$

$$= \frac{1}{a^2+\omega^2}$$

$$\frac{1}{a^2-(j\omega)^2} = \frac{1}{(a+j\omega)(a-j\omega)}$$

$$= \frac{A}{a+j\omega} + \frac{B}{a-j\omega} = \frac{A}{j\omega+a} - \frac{B}{j\omega-a}$$

$$1 = A(j\omega - a) - B(j\omega + a).$$

Put $j\omega = a \Rightarrow 1 = -2a \cdot B$, $B = \frac{-1}{2a}$.

Put $j\omega = -a \Rightarrow 1 = -2aA$, $A = \frac{-1}{2a}$.

$$|X(\omega)|^2 = \frac{-1}{2a} \cdot \frac{1}{a+j\omega} + \frac{1}{2a} \cdot \frac{1}{a-j\omega}.$$

Taking inv:

$$r_{xx}(t) = \frac{-1}{2a} e^{-at}u(t) + \frac{1}{2a} e^{at}u(t)$$

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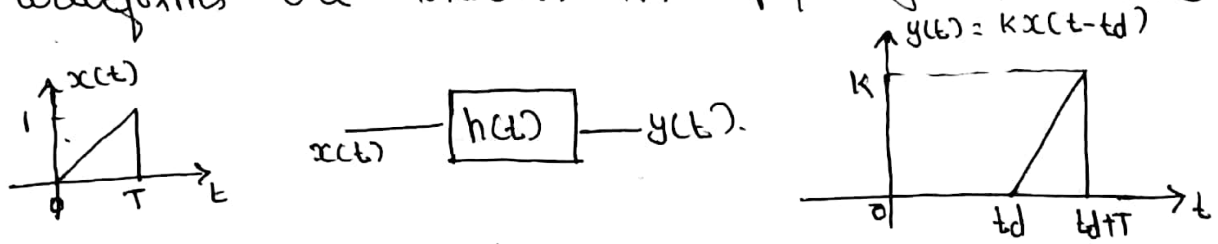
Distortionless transmission through a system:

The change of shape of the signal when it is transmitted through a s/m is called distortion. Transmission of a signal through a system is said to be distortionless if the o/p is an exact replica of the i/p signal. This replica may have different magnitude and also it may have different time delay. A constant change in magnitude and a constant time delay are not considered as distortion. Only the shape of the signal is important. Mathematically we can say that a signal $x(t)$ is transmitted without distortion if the output

$$y(t) = kx(t-t_d) \rightarrow ①$$

where k is a constant representing the change in amplitude (amplification or attenuation) at t_d is delay time. A distortionless s/m and typical i/p and o/p waveforms are shown in fig. given below



Taking FT on both sides of the eqn ①

$$Y(\omega) = ke^{-j\omega t_d} X(\omega) \quad \text{(By shifting property)}$$

Therefore, for distortionless transmission, the transfer function of the s/m must be of the form

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = ke^{-j\omega t_d}.$$

Taking inverse FT, the corresponding impulse response must be $h(t) = k\delta(t-t_d)$.

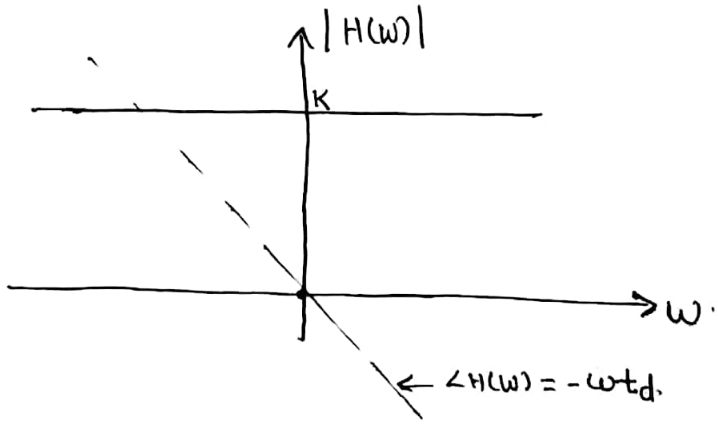
It is clear that the magnitude of the transfer fun^n.

$|H(\omega)| = K$ and that it is a constant for all values of ω.

The phase shift $\angle H(\omega) = -\omega t_d$ and it varies linearly with frequency, In general $\angle H(\omega) = n\pi - \omega t_d$

So for distortionless transmission of a signal through a s/m, * the magnitude $|H(\omega)|$ should be a constant. ie all the frequency components of the i/p signal must undergo the same amount of amplification and attenuation. * phase spectrum should be proportional to frequency.

The magnitude and phase characteristics of a distortionless transmission system is shown in fig. given below.



Transmission of a rectangular pulse through an ideal low pass filter:

An ideal filter has very sharp cutoff characteristic, and it passes signal of certain specified band of frequencies exactly and totally rejects signals of frequencies outside the band.

The frequency response of an ideal LPF with cut off frequency, ω_c is defined by

$$H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The impulse response of the filter is

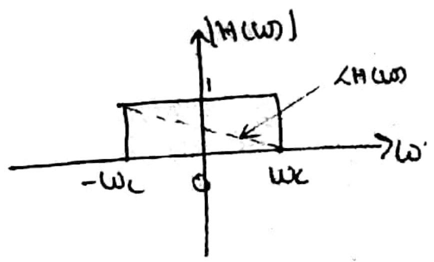
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega t_0}\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)}\, d\omega$$

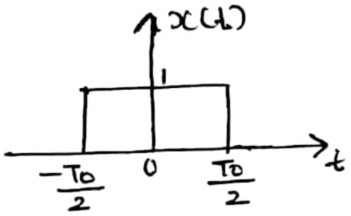
$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(t-t_0)}}{j(t-t_0)}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi j(t-t_0)}\left[e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}\right]$$

$$= \frac{1}{\pi(t-t_0)}\left[\frac{e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}}{2j}\right]$$

$$= \frac{1}{\pi}\,\frac{\sin\omega_c(t-t_0)}{t-t_0} = \frac{\omega_c}{\pi}\,\frac{\sin\omega_c(t-t_0)}{\omega_c(t-t_0)} = h(t).$$



Rectangular pulse:



$$x(t) = \begin{cases} 1 & |t| \le \frac{T_0}{2} \\ 0 & |t| > \frac{T_0}{2} \end{cases}$$

o/p of the LTI s/m, $y(t) = x(t) * h(t)$.

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-T_0/2}^{T_0/2} \frac{\omega_c}{\pi} \frac{\sin \omega_c (t-t_0-\tau)}{\omega_c (t-t_0-\tau)}\, d\tau .$$

Put $\lambda = \omega_c (t - t_0 - \tau)$

$d\lambda = -\omega_c\, d\tau . \Rightarrow d\tau = -\frac{d\lambda}{\omega_c}.$

$\tau \to -\frac{T_0}{2} \Rightarrow \lambda \to \omega_c (t - t_0 + \frac{T_0}{2})$
$\to a.$

$\tau \to \frac{T_0}{2} \Rightarrow \lambda \to \omega_c (t - t_0 - \frac{T_0}{2})$
$\to b$

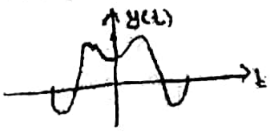
$$\therefore y(t) = \frac{\omega_c}{\pi} \int_a^b \frac{\sin\lambda}{\lambda} (-) \frac{d\lambda}{\omega_c} = \frac{1}{\pi} \int_a^b \frac{-\sin\lambda}{\lambda}\, d\lambda.$$

$$= \frac{1}{\pi}\left[\int_0^a \frac{\sin\lambda}{\lambda}\, d\lambda - \int_0^b \frac{\sin\lambda}{d\lambda}\, d\lambda \right]$$

$$y(t) = \frac{1}{\pi}\left[S_i(a) - S_i(b) \right]$$

The relationship exist b/w 2 parameters. a) duration of rectangular i/p pulse T_0 and b) cut off freq. of filter ω_c.

when $\omega_c > \frac{2\pi}{T_0}$.



when $\omega_c = \frac{2\pi}{T_0}$

when $\omega_c < \frac{2\pi}{T_0}$

Hilbert transform:

* When the phase angles of all the positive frequency spectral components of a given signal are shifted by -90° and the phase angles of all the negative frequency spectral components are shifted by $+90^\circ$, the resulting function of time is called Hilbert transform of the signal.
* The amplitude spectrum of the signal is unchanged by Hilbert transform operation. Only the phase spectrum of the signal is changed.
* The Hilbert transformed signal is also a time domain signal.

$$x(t) \longrightarrow \boxed{\text{Hilbert transformer}} \longrightarrow \hat{x}(t)$$

$x(t)$ (time function) — $\hat{x}(t)$ (time function)

$x(t)$ is the i/p to the Hilbert transform and $\hat{x}(t)$ is its o/p.

The impulse response of Hilbert transform is $h(t) = \frac{1}{\pi t}$

$\therefore$ the o/p, $\hat{x}(t) = x(t) * h(t)$.

$$= x(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} x(\tau) \cdot \frac{1}{\pi(t-\tau)} d\tau.$$

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

The inverse Hilbert transform, by means of which the original signal $x(t)$ is recovered from $\hat{x}(t)$ is defined by

$$x(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} d\tau$$

The functions $x(t)$ and $\hat{x}(t)$ are said to be Hilbert transform pair.

For time function $\frac{1}{\pi t}$, we have $\frac{1}{\pi t} \xleftrightarrow{FT} -j\,\mathrm{sgn}(\omega)$

where $\mathrm{sgn}(\omega)$ is the signum function in the frequency domain is given by $\mathrm{sgn}(\omega) = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$

W.K.T

$$\hat{x}(t) = x(t) * h(t).$$
$$= x(t) * \frac{1}{\pi t}$$

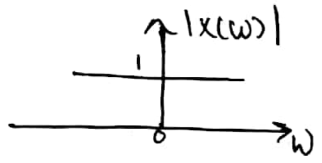
Taking FT

$$\therefore \hat{X}(\omega) = X(\omega) . * -j\, sgn(\omega)$$

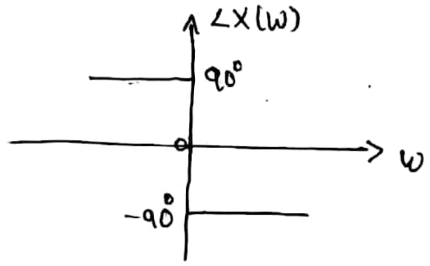
$$\hat{X}(\omega) = -j\, sgn(\omega)\, X(\omega).$$

This implies that $\hat{X}(\omega) = \begin{cases} -j X(\omega) & \omega > 0 \\ j X(\omega) & \omega < 0 \end{cases}$

Since $\hat{X}(\omega)$ is the spectrum of $\hat{x}(t)$ and $X(\omega)$ is the spectrum of $x(t)$, this device may be considered as one that produces a phase shift of $-90°$ for all positive frequencies of the i/p signal and $+90°$ for all negative frequencies as shown in fig. given below.



magnitude response



Properties of Hilbert transform:

1) It does not change the domain of a signal
2) It does not alter the amplitude spectrum of a signal
3) A signal $x(t)$ and its Hilbert transform $\hat{x}(t)$ are orthogonal to each other i.e $\int_{-\infty}^{\infty} x(t)\,\hat{x}(t)\,dt = 0.$
4) If $\hat{x}(t)$ is the Hilbert transform of $x(t)$, the Hilbert transform of $\hat{x}(t)$ is $-x(t)$.

Applications:

1) To realize phase selectivity in the generation of single side band modulation systems
2) To represent band pass signals.

8) To relate the gain and phase characteristics of linear communication channels and filters of ~~[illegible]~~

Qn. Find the Hilbert transform of $x(t) = \sin \omega_0 t$

$$X(\omega) = -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$\hat{X}(\omega) = -j\,\mathrm{sgn}(\omega) \cdot X(\omega).$$

$$= -j\left\{-j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]\right\}\mathrm{sgn}(\omega).$$

$$= -\pi\left[\delta(\omega-\omega_0) \cancel{-} + \delta(\omega+\omega_0)\right]\mathrm{sgn}(\omega)$$

$1 \xleftrightarrow{CTFT} 2\pi\delta(\omega).$

$$= -\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

Taking inv. FT

$$\hat{x}(t) = -\cos\omega_0 t$$

Qn. Find the Hilbert transform of $x(t) = \cos\omega_0 t$

$$X(\omega) = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$2\pi\delta(\omega)$

$$\hat{x}(\omega) = -j\,\mathrm{sgn}(\omega) \cdot X(\omega)$$

$$= -j\left\{\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]\right\}\mathrm{sgn}(\omega)$$

$$= -j\left\{\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]\right\}$$

$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt$

$\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt$

$$= -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

~~$= \sin\omega_0 t$~~

Taking inv. FT

$$\hat{x}(t) = \sin\omega_0 t.$$

$x(t) = \sin\omega_0 t$

$$= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

~~[illegible]~~

~~$X(\omega) = \frac{1}{2j}\left[FT\{e^{j\omega_0 t}\} - FT\{e^{-j\omega_0 t}\}\right]$~~

$$= \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$= -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

Laplace transform:

It is used for the analysis of continuous time signals and systems. The laplace transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the begining of the process itself. To solve differential eqns. which are in time domain, they are first converted into algebraic eqns in frequency domain using laplace transform, the algebraic equations are manipulated in s-domain and the result obtained in frequency domain is converted back into time domain using inverse laplace transform.

The bilateral laplace transform of a continuous time signal $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

The inverse laplace transform of $X(s)$ is defined as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

The unilateral laplace transform of a continuous time signal $x(t)$ is defined as.

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt.$$

Region of Convergence (ROC)

The range of σ (real part of s) for which the laplace transform converge is called Region of convergence (ROC).

Properties of ROC:

1) The ROC of X(s) consists of parallel strips to the Imaginary axis.
2) The ROC of LT does not include any poles of X(s).
3) If x(t) is a finite duration signal and is absolutely integrable then the ROC of X(s) is the entire s-plane.
4) For the right sided (causal) signal if the Re(s) = σ_0 and is in ROC, then for all the values of s for which Re(s) > σ_0 is also in ROC.
5) If x(t) is a left sided (non-causal) signal and if Re(s) = σ_0 is in ROC, then for all values of s for which Re(s) < σ_0 is also in ROC.
6) If x(t) is two sided signal and if Re(s) = σ_0 is in ROC, then the ROC of X(s) will consist of a strip in the s-plane which will include Re(s) = σ_0.

* The points in the s-plane at which X(s) = ∞ are called poles & the points in the s-plane at which X(s) = 0 are called zeros of X(s).

Qn. Determine the laplace transform, ROC and location of poles of the signal $x(t) = e^{-at}u(t)$.

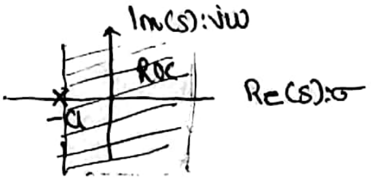
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-at} e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+s)t}\, dt \qquad \mathrm{Re}(s+a) > 0, \quad \mathrm{Re}(s) > -a$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)} \right]_0^{\infty} = \frac{-1}{s+a}\left[e^{-\infty} - e^{0} \right]$$

$$= \frac{1}{s+a}$$ no zeros, poles $s = -a$



Qn. Find LT, ROC and poles and zeros of $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$

$$x(t) = \underbrace{e^{-at}u(t)}_{x_1(t)} + \underbrace{e^{-bt}u(-t)}_{x_2(t)} \quad , \; b > a$$

$X(s) = X_1(s) + X_2(s)$.

$x_1(t) = e^{-at}u(t) \Longleftrightarrow X_1(s) = \frac{1}{s+a} \qquad \sigma > -a$

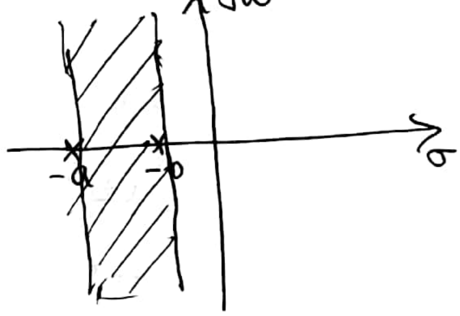
$x_2(t) = e^{-bt}u(-t) \longrightarrow X_2(s) = \frac{-1}{s+b} \qquad \sigma < -b$

$X(s) = X_1(s) + X_2(s) = \frac{1}{s+a} - \frac{1}{s+b} = \frac{\cancel{s}+b-\cancel{s}-a}{(s+a)(s+b)}$

$= \frac{b-a}{(s+a)(s+b)}$.

$\therefore$ ROC: $-a < \sigma < -b$

no zeros, poles: $s = -a$ and $s = -b$

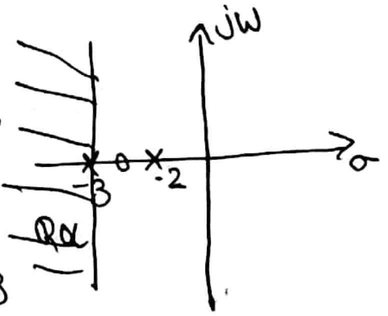


Qn. Determine the LT of $x(t) = e^{-2t}u(-t) + e^{-3t}u(-t)$. Locate the poles and zero of $X(s)$ and also the ROC in the s-plane.

$x_1(t) = e^{-2t}u(-t) \Rightarrow X_1(s) = -\frac{1}{s+2}$, $\sigma < -2$; $x_2(t) = e^{-3t}u(-t) \Rightarrow X_2(s) = \frac{-1}{s+3}$, $\sigma < -3$.

$X(s) = \frac{-1}{s+2} - \frac{1}{s+3} = \frac{-(s+3)-(s+2)}{(s+2)(s+3)}$

$= \frac{-s-3-s-2}{(s+2)(s+3)} = \frac{-2s-5}{(s+2)(s+3)}$.



$\therefore$ ROC: $\sigma < -3$.

zeros: $s = -2.5$ poles $s = -2$ & $s = -3$

Relation between LT and FT:

LT of $x(t) = X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-St}\, dt$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

If $\sigma = 0$, then $X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt =$ FT of $x(t)$.

Relation between Z-transform and laplace transform

Let $x(t)$ be a continuous time signal. The discrete time signal $x^*(t)$ can be obtained by sampling $x(t)$ with sampling period of T sec. ie. $x^*(t)$ is obtained by multiplying $x(t)$ with a seq. of impulse T sec. apart

$$x^*(t) = \sum_{n=0}^{\infty} x(nT)\, \delta(t - nT).$$

The laplace transform of $x^*(t)$ is given by

$$L\{x^*(t)\} = X^*(S) = L\left[\sum_{n=0}^{\infty} x(nT)\, \delta(t - nT)\right]$$

$$= \sum_{n=0}^{\infty} x(nT)\, L\{\delta(t-nT)\} = \sum_{n=0}^{\infty} x(nT)\, e^{-nTs} \rightarrow ①$$

~~The z-transform of x(nT) is given by~~.

$$\cancel{Z\{x(nT)\} = \sum_{n=0}^{\infty} x(nT)\, z^{-n}} \rightarrow ②$$

Put $z = e^{TS}$ in eqn ① we get the Z-transform of $x(nT)$.

$$\therefore L\{x^*(t)\} = \sum_{n=0}^{\infty} x(nT)\, z^{-n} = Z\{x(nT)\}$$

Ex. Find the Laplace transform of $x(t) = \cos\omega t\, u(t)$.

$$\cancel{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt} = \cancel{\int_{-\infty}^{\infty} \cos\omega t\, u(t) e^{-st} dt}$$

$$\cancel{= \int_{0}^{\infty} \cos\omega t\, e^{-st} dt}$$

$$x(t) = \underbrace{\frac{1}{2} e^{j\omega t} u(t)}_{x_1(t)} + \underbrace{\frac{1}{2} e^{-j\omega t} u(t)}_{x_2(t)}.$$

$$X(s) = X_1(s) + X_2(s)$$

$$X_1(s) = \frac{1}{2} \frac{1}{s - j\omega} \quad \text{and} \quad X_2(s) = \frac{1}{2} \frac{1}{s + j\omega}.$$

$$X(s) = \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right] = \frac{1}{2}\left[\frac{s + j\omega + s - j\omega}{s^2 + \omega^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 + \omega^2}\right] = \frac{s}{s^2 + \omega^2}$$

$$\boxed{\cos\omega t\, u(t) \overset{L}{\longleftrightarrow} \frac{s}{s^2 + \omega^2}}$$

Ex. Find the LT of $x(t) = \sin\omega t\, u(t)$.

$$x(t) = \sin\omega t\, u(t) = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right] u(t)$$

$$= \frac{1}{2j}\left[e^{j\omega t} u(t) - e^{-j\omega t} u(t)\right].$$

$$X(s) = \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right] = \frac{1}{2j}\left[\frac{\cancel{s} + j\omega - \cancel{s} + j\omega}{s^2 + \omega^2}\right]$$

$$= \frac{1}{\cancel{2j}}\left[\frac{\cancel{2j}\,\omega}{s^2 + \omega^2}\right] = \frac{\omega}{s^2 + \omega^2}$$

$$\boxed{\sin\omega t\, u(t) \overset{L}{\longleftrightarrow} \frac{\omega}{s^2 + \omega^2}}$$

Properties of laplace transform:

1) linearity:

$$a x_1(t) + b x_2(t) \overset{L}{\longleftrightarrow} a X_1(s) + b X_2(s).$$

2) Time shifting:

$$x(t-t_0) \overset{L}{\longleftrightarrow} e^{-st_0} X(s).$$

3) frequency shifting:

$$e^{at} x(t) \overset{L}{\longleftrightarrow} X(s-a).$$

4) Time scaling:

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right)$$

5) frequency scaling:

$$\frac{1}{a} x(t/a) \overset{L}{\longleftrightarrow} X(as).$$

6) Time differentiation:

$$\frac{d}{dt} x(t) \overset{L}{\longleftrightarrow} s X(s) - x(0).$$

7) Time integration:

$$\int_0^t x(\tau) d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s}$$

8) Time convolution:

$$x_1(t) * x_2(t) \overset{L}{\longleftrightarrow} X_1(s) . X_2(s).$$

9) Conjugation:

$$x^*(t) \overset{L}{\longleftrightarrow} X^*(-s).$$

10) complex frequency differentiation:

$$-t\, x(t) \overset{L}{\longleftrightarrow} \frac{d}{ds} X(s).$$

$$t^n x(t) \overset{L}{\longleftrightarrow} (-1)^n \frac{d^n}{ds^n} X(s).$$

11) Initial value theorem:

$$x(0) = \underset{s \to \infty}{Lt}\, s X(s).$$

12) Final value theorem:

$$x(\infty) = \underset{s \to 0}{Lt}\, s X(s).$$

3

Qn. Find the unilateral laplace transform of the following signals.

a) ~~...~~. $x(t) = \delta(t)$.

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} \delta(t)\, e^{-st}\, dt$$

$$= e^{-st}\Big|_{t=0} = 1$$

b) $x(t) = 1$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{0}^{\infty}$$

$$= -\frac{1}{s}\left[0 - 1\right] = \underline{\underline{1/s}}$$

c) $x(t) = t$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$= \left[t\, \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s}\, dt\right]_{0}^{\infty}$$

$$= \left[t\, \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_{0}^{\infty}$$

$$= \left[\infty \frac{e^{-\infty}}{-s} - \frac{e^{-\infty}}{s^2} - 0 + \frac{e^{0}}{s^2}\right]$$

$$= \left[0 - 0 - 0 + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

d) $x(t) = t^2$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} t^2 e^{-st}\, dt$$

$$= \left[t^2 \frac{e^{-st}}{-s} - \int 2t\, \frac{e^{-st}}{-s}\, dt\right]_{0}^{\infty}$$

$$= \left[t^2 \frac{e^{-st}}{-s} - \left(2t\, \frac{e^{-st}}{s^2} - \int 2 \cdot \frac{e^{-st}}{+s^2}\, dt\right]_{0}\right.$$

$$= \left[t^2 \frac{e^{-st}}{s} - \left(2t \frac{e^{-st}}{s^2} - 2 \frac{e^{-st}}{-s^3} \right) \right]_0^\infty$$

$$= \left[t^2 \frac{e^{-st}}{s} - 2t\frac{e^{-st}}{s^2} - 2\frac{e^{-st}}{s^3} \right]_0^\infty$$

$$= \left[\infty \frac{e^{-\infty}}{s} - 2\infty \frac{e^{-\infty}}{s^2} - 2\frac{e^{-\infty}}{s^3} - 0 + 0 + \frac{2e^0}{s^3} \right]$$

$$= \left[0 - 0 - 0 - 0 + 0 + \frac{2}{s^3} \right] = \frac{2}{s^3}$$

$$\boxed{\begin{aligned} t &\overset{L}{\longleftrightarrow} \frac{1}{s^2} \\ t^2 &\overset{L}{\longleftrightarrow} \frac{2}{s^3} \\ t^n &\overset{L}{\longleftrightarrow} \frac{n!}{s^{n+1}} \end{aligned}}$$

Laplace transform of some standard signals:

$x(t)$	$X(s)$
$\delta(t)$	1
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$t e^{-at}$	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$x(t)$	$X(s)$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Qn. Find the LT of $x(t) = u(t-2)$.

$$u(t) \xleftrightarrow{L} \frac{1}{S}$$

By shifting property $x(t-t_0) \xleftrightarrow{L} e^{-St_0} X(S)$

$$\therefore u(t-2) \xleftrightarrow{L} \frac{e^{-2S}}{S}$$

Qn. Find the LT of $t^2 e^{-2t} u(t)$.

$$e^{-2t} u(t) \xleftrightarrow{L} \frac{1}{S+2}$$

By complex freq. differentiation property

$$t^n x(t) \xleftrightarrow{L} (-1)^n \frac{d^n}{ds^n} X(S).$$

$$t^2 e^{-2t} u(t) \xleftrightarrow{L} (-1)^2 \frac{d^2}{ds^2} \cdot \frac{1}{S+2}.$$

$$= \frac{d^2}{dS^2} \frac{1}{S+2}$$

$$= \frac{d}{dS} \left[\frac{(S+2)\cdot 0 - (1\cdot 1)}{(S+2)^2} \right]$$

$$= \frac{d}{dS} \left[\frac{-1}{(S+2)^2} \right]$$

$$= \frac{(S+2)^2 \cdot 0 - -1\ 2(S+2)}{(S+2)^4}$$

$$= \frac{2(S+2)}{(S+2)^4} = \frac{2}{(S+2)^3}$$

Qn. For the following transform pair $L\{x(t)\} = \frac{2S}{S^2-2}$. determine the LT of $x(2t)$.

By time scaling property: $x(at) \xleftrightarrow{L} \frac{1}{a} X\left(\frac{S}{a}\right)$

$$\therefore x(2t) \xleftrightarrow{L} \frac{1}{2} \frac{2(S/2)}{(S/2)^2 - 2} = \frac{1}{2} \frac{S}{\frac{S^2}{4} - 2} = \frac{1}{\cancel{2}} \left[\frac{S}{\frac{S^2-8}{\cancel{4}2}} \right]$$

$$= \frac{2S}{S^2-8}$$

Qn. Find the LT of $x(t) = e^{-2t} \sin 2t \, u(t)$

W.K.T: $\sin \omega t \, u(t) \overset{L}{\longleftrightarrow} \frac{\omega}{s^2+\omega^2}$

$\sin 2t \, u(t) \overset{L}{\longleftrightarrow} \frac{2}{s^2+4}$

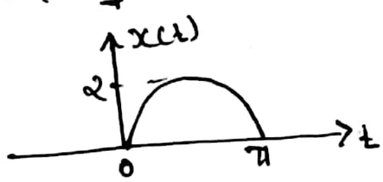
Using frequency shift property:

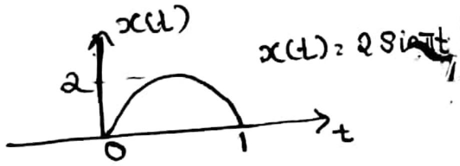
$e^{at} x(t) \overset{L}{\longleftrightarrow} X(s-a)$

$e^{-at} x(t) \overset{L}{\longleftrightarrow} X(s+a)$

$\therefore \; e^{-2t} \sin 2t \, u(t) \longleftrightarrow \frac{2}{(s+2)^2+4} = \underline{\underline{\frac{2}{s^2+4s+8}}}$

Qn. Determine LT of





Ans:

from the fig: $x(t) = 2\sin t \quad 0 \le t \le \pi$

$= 0 \quad 0 \; t > \pi$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{0}^{\pi} 2 \sin t \, e^{-st} dt$$

$$= 2 \int_0^{\pi} \sin t \, e^{-st} dt$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \int \cos t \frac{e^{-st}}{-s} dt \right]_0^{\pi}$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \left[\cos t \frac{e^{-st}}{s^2} - \int -\sin t \frac{e^{-st}}{s^2} dt \right] \right]_0^{\pi}$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \frac{1}{s^2} \left[\cos t \, e^{-st} + \int \sin t \, e^{-st} dt \right] \right]_0^{\pi}$$

$$= 2 \left[\sin t \frac{e^{-st}}{-s} - \frac{1}{s^2} \underbrace{\cos t}_{\cos \pi = -1} e^{-st} \right]_0^{\pi} - \frac{2}{s^2} \int_0^{\pi} \sin t \, e^{-st} dt$$

$$= 2 \left[0 - 0 + \frac{1}{s^2} e^{-\pi s} + \frac{1}{s^2} \right] - \frac{X(s)}{s^2}.$$

$$X(S) = \frac{2}{S^2}\left[1+e^{-\pi S}\right] - \frac{X(S)}{S^2}.$$

$$X(S) + \frac{X(S)}{S^2} = \frac{2}{S^2}\left[1+e^{-\pi S}\right]$$

$$X(S)\left[1+\frac{1}{S^2}\right] = \frac{2}{S^2}\left[1+e^{-\pi S}\right].$$

$$X(S)\left[\frac{S^2+1}{S^2}\right] = \frac{2}{S^2}\left[1+e^{-\pi S}\right]$$

$$X(S) = \frac{2\left[1+e^{-\pi S}\right]}{S^2+1} \quad //$$

Alternative method:

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-St}\, dt = 2\int_0^{\pi} \sin t\, e^{-St}\, dt = 2\int_0^{\pi} e^{-St} \sin t\, dt$$

W.K.T $\int e^{at} \sin t\, dt = \frac{e^{at}}{a^2+1}(a \sin t - \cos t).$

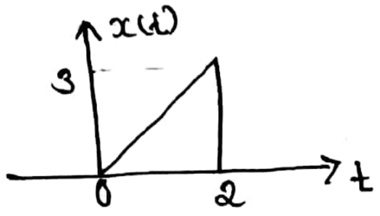
$$\therefore 2\int_0^{\pi} e^{-St} \sin t\, dt = 2\left[\frac{e^{-St}}{S^2+1}(-S\sin t - \cos t)\right]_0^{\pi}$$

$$= 2\left[\frac{e^{-\pi S}}{S^2+1}(1) - \frac{1}{S^2+1}(-1)\right]$$

$$= 2\left[\frac{e^{-\pi S}}{S^2+1} + \frac{1}{S^2+1}\right]$$

$$= \frac{2\left[e^{-\pi S}+1\right]}{S^2+1}$$

Qn. Determine the LT of the saw tooth waveform shown in fig. given below.



Ans:

$$x(t) = \begin{cases} \frac{3}{2}t, & 0 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_0^2 \frac{3}{2} t\, e^{-st}\, dt = \frac{3}{2}\int_0^2 t\, e^{-st} dt$$

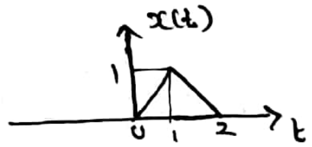
$$= \frac{3}{2}\left[t\,\frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s}\, dt \right]_0^2$$

$$= \frac{3}{2}\left[t\,\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^2 = \frac{3}{2}\left[\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} - \left(-\frac{1}{s^2}\right) \right]$$

$$= \frac{3}{2} \cdot \frac{2e^{-2s}}{-s} - \frac{3}{2}\,\frac{e^{-2s}}{s^2} + \frac{3}{2}\,\frac{1}{s^2}.$$

$$= \frac{3}{2}\,\frac{1}{s^2} - e^{-2s}\left[\frac{3}{s} + \frac{3}{2s^2}\right]$$

Qn. Determine the LT of



$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_0^1 t\, e^{-st}\, dt + \int_1^2 (2-t)\, e^{-st}\, dt$$

$$= \left[t\,\frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s}\, dt \right]_0^1 + \left[(2-t)\,\frac{e^{-st}}{-s} - \int (-1)\, \frac{e^{-st}}{-s} \right]_1^2$$

$$= \left[-t\,\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^1 + \left[-(2-t)\,\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_1^2$$

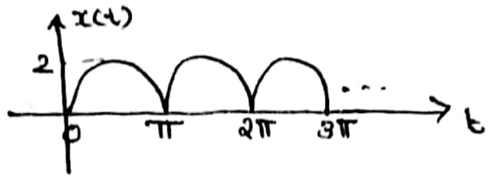
$$= -\cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2}$$

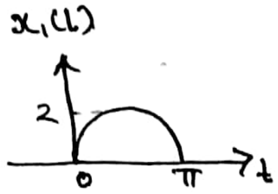
$$= -\frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} + \frac{1}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2} = \left[\frac{1 - e^{-s}}{s}\right]^2$$

Laplace transform of periodic functions.

$$X(s) = \frac{1}{1-e^{-sT}} X_1(s) \quad , \text{ where } T \to \text{period.}$$

Qn. Determine the LT of of a full wave rectifier.



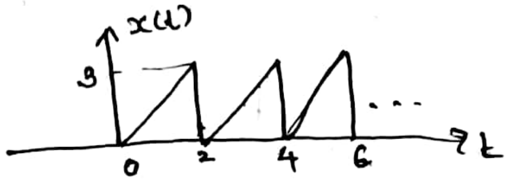


$T = \pi$

$$X_1(s) = \frac{2\left[e^{-\pi s}+1\right]}{s^2+1}.$$

$$X(s) = \frac{1}{1-e^{-\pi s}} \cdot \frac{2\left[e^{-\pi s}+1\right]}{s^2+1} = \frac{2\left[e^{-\pi s}+1\right]}{\left[1-e^{-\pi s}\right]\left[s^2+1\right]}.$$

Qn. Determine the LT of



$T = 2$

$$X_1(s) = \frac{3}{2}\frac{1}{s^2} - e^{-2s}\left[\frac{3}{s}+\frac{3}{2s^2}\right]$$

$$X(s) = \left\{\frac{3}{2}\frac{1}{s^2} - e^{-2s}\left[\frac{3}{s}+\frac{3}{2s^2}\right]\right\}\frac{1}{1-e^{-2s}}.$$

$$= \frac{3}{2}\frac{1}{s^2(1-e^{-2s})} - \frac{e^{-2s}}{1-e^{-2s}}\left[\frac{3}{s}+\frac{3}{2s^2}\right] //.$$

Qn. Find the LT of $x(t) = t e^{-2t} \sin 2t\, u(t)$ using properties of LT.

$$x_1(t) = \sin 2t\, u(t) \xleftrightarrow{L} X_1(s) = \frac{2}{s^2+4}$$

$$x_2(t) = e^{-2t} \sin 2t\, u(t) \xleftrightarrow{L} X_2(s) = \frac{2}{(s+2)^2+4} = \frac{2}{s^2+4s+8}$$

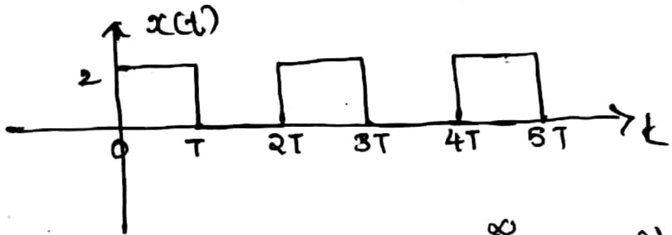
$$x(t) = t e^{-2t} \sin 2t\, u(t) \xleftrightarrow{L} X(s) = -\frac{d}{ds} \frac{2}{s^2+4s+8}$$

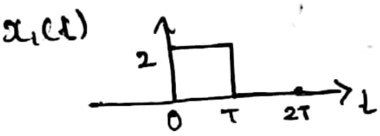
(Complex freq. differentiation property)

$$= -\left[\frac{(s^2+4s+8)\cdot 0 - 2(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= +\frac{2(2s+4)}{(s^2+4s+8)^2} = \frac{4(s+2)}{(s^2+4s+8)^2}$$

Qn. Find the LT of the waveform.





$$X_1(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt = \int_0^T 2 e^{-st}\, dt$$

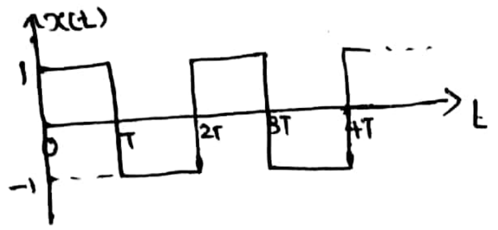
$$= 2\left[\frac{e^{-st}}{-s}\right]_0^T = \frac{2}{s}\left[1 - e^{-sT}\right]$$

$$X(s) = \frac{1}{(1-e^{-2sT})} \cdot \frac{2}{s}\left[1 - e^{-sT}\right]$$

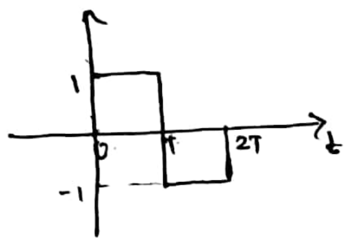
$$= \frac{2}{s} \frac{\cancel{(1-e^{-sT})}}{\cancel{(1-e^{-sT})}(1+e^{-sT})}$$

$$= \frac{2}{s}\left[\frac{1}{1+e^{-sT}}\right]$$

Qn. Find the LT of the waveform



Soln: $x_1(t)$



$T = 2T$ $\qquad\qquad u(t-T) = e^{-sT}\frac{1}{s}$

mathematically $x_1(t) = u(t) - 2u(t-T) + u(t-2T)$.

$$\downarrow L \qquad \downarrow L$$

$$\frac{1}{s} - 2\frac{e^{-sT}}{s} + \frac{e^{-2sT}}{s}$$

$$\therefore X_1(s) = \frac{1}{s} - 2\frac{e^{-sT}}{s} + \frac{e^{-2s}}{s} = \frac{1 - 2e^{-sT} + e^{-2sT}}{s} = \frac{\left[1 - e^{-sT}\right]^2}{s}$$

$$X(s) = \frac{1}{\left[1 - e^{-2sT}\right]} \cdot \frac{1}{s}\left[1 - e^{-sT}\right]^2$$

$$= \frac{1}{s} \cdot \frac{\cancel{\left[1 - e^{-sT}\right]}\left[1 - e^{-sT}\right]}{\cancel{\left[1 - e^{-sT}\right]}\left[1 + e^{-sT}\right]}$$

$$= \frac{1}{s}\frac{\left[1 - e^{-sT}\right]}{\left[1 + e^{-sT}\right]}$$

Inverse Laplace transform:

The inverse LT of $X(s)$ is defined as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \, ds$$

Qn. Find the inverse LT of $X(s) = \frac{s}{s^2 + 5s + 6}$ (partial fraction method).

$$X(s) = \frac{s}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}.$$

$$s = A(s+3) + B(s+2)$$

Put $s = -2 \Rightarrow A = -2$ Put $s = -3 \Rightarrow B = 3$

$$\therefore X(s) = -2 \frac{1}{s+2} + 3 \frac{1}{s+3}.$$

Taking inverse LT

$$x(t) = -2 e^{-2t} u(t) + 3 e^{-3t} u(t)$$

Qn. Find the inverse LT of $X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s^2+2s+1)}$

$$X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$

$A = 2$, $B = 1$, $C = 1$

$$\therefore X(s) = 2 \frac{1}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

Taking inv. LT.

$$x(t) = 2 e^{-2t} u(t) + e^{-t} u(t) + t e^{-t} u(t)$$

Qn. Find the inv. LT of $X(s) = \frac{2s+1}{(s+1)(s^2+2s+2)}$

$$X(s) = \frac{2s+1}{(s+1)(s-(-1+j))(s-(-1-j))} = \frac{A}{(s+1)} + \frac{B}{(s-(-1+j))} + \frac{C}{(s-(-1-j))}$$

$$2s+1 = A(s-(-1+j))(s-(-1-j)) + B(s+1)(s-(-1-j)) + C(s+1)(s-(-1+j))$$

Put $s=-1 \Rightarrow -1 = A(\cancel{-1}+\cancel{1}-j)(\cancel{-1}+\cancel{1}+j)$.

$$A = \underline{\underline{-1}}$$

Put $s=(-1+j) \Rightarrow$ $2(-1+j)+1 = B(-1+j+1)(-1+j+1+j)$.

$$-2+2j+1 = B(j)(2j).$$

$$2j-1 = -2B \Rightarrow B = \frac{2j-1}{-2} = \underline{0.5-j}$$

Put $s=(-1-j) \Rightarrow$ $2(-1-j)+1 = C(-1-j+1)(-1-j+\cancel{1}-j)$.

$$-2-2j+1 = C(-j)(-2j). \Rightarrow -1-2j = -2C$$

$$C = \underline{0.5+j}$$

$$\therefore X(s) = \frac{-1}{s+1} + (0.5-j)\frac{1}{s-(-1+j)} + (0.5+j)\frac{1}{s-(-1-j)}.$$

$$= -e^{-t}u(t) + (0.5-j)\,e^{(-1+j)t}u(t) + (0.5+j)\,e^{(-1-j)t}u(t)$$

$$= -e^{-t}u(t) + (0.5-j)\,e^{-t}.e^{+jt}.u(t) + (0.5+j)\,e^{-t}e^{-jt}u(t)$$

$$= -e^{-t}u(t) + \underbrace{0.5e^{-t}e^{jt}u(t)} - je^{-t}e^{jt}u(t) + \underbrace{0.5e^{-t}e^{-jt}u(t)} + je^{-t}e^{-jt}u(t)$$

$$= -e^{-t}u(t) + 0.5e^{-t}(e^{jt}+e^{-jt})u(t) - 2j \cdot je^{-t}\frac{(e^{jt}-e^{-jt})}{2j}u(t)$$

$$= -e^{-t}u(t) + e^{-t}\cos t\, u(t) + 2e^{-t}\sin t\, u(t)$$

$$= -e^{-t}u(t) + e^{-t}(\cos t + 2\sin t)u(t)$$

Qn. Find the inverse LT of $X(s) = \dfrac{2}{(s+4)(s-1)}$

if the ROC is a) $-4 < Re(s) < 1$ b) $Re(s) > 1$

c) $Re(s) < -4$

$$X(s) = \frac{2}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1}$$

$2 = A(s-1) + B(s+4)$.

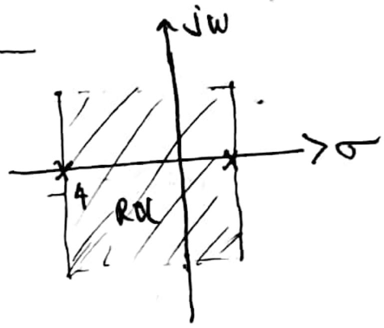
Put $s = -4 \Rightarrow 2 = -5A \Rightarrow A = -2/5$

Put $s = 1 \Rightarrow 2 = 5B \Rightarrow B = 2/5$.

$$\therefore X(s) = -\frac{2}{5}\frac{1}{s+4} + \frac{2}{5}\frac{1}{s-1}$$

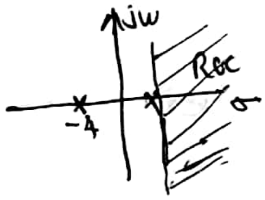
a) If the ROC is $\underbrace{-4 < Re(s)}_{\text{right}} \overbrace{< 1}^{\text{left}}$

$$\therefore x(t) = \frac{-2}{5}e^{-4t}u(t) - \frac{2}{5}e^{t}u(-t)$$



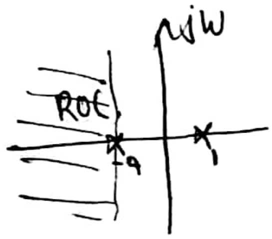
b) If ROC is $Re(s) > 1$

$$X(s) = \frac{-2}{5}\underbrace{\frac{1}{s+4}}_{\text{causal}} + \frac{2}{5}\underbrace{\frac{1}{s-1}}_{\text{causal}}.$$



$$\therefore x(t) = \frac{-2}{5}e^{-4t}u(t) + \frac{2}{5}e^{t}u(t)$$

c) If the ROC is $Re(s) < -4$



$$X(s) = \underbrace{-\frac{2}{5}\frac{1}{s+4}}_{\text{non causal}} + \underbrace{\frac{2}{5}\frac{1}{s-1}}_{\text{non causal}}.$$

$$\therefore x(t) = \frac{2}{5}e^{-4t}u(-t) - \frac{2}{5}e^{t}u(-t).$$

Qn. Use the convolution theorem of LT to find $y(t) = x_1(t) * x_2(t)$ when $x_1(t) = e^{-3t}u(t)$ & $x_2(t) = u(t-2)$

$$y(t) = x_1(t) * x_2(t) \xleftrightarrow{L} Y(S) = X_1(S) \cdot X_2(S).$$

$$x_1(t) = e^{-3t}u(t) \Rightarrow X_1(S) = \frac{1}{S+3}.$$

$$x_2(t) = \cancel{e}\, u(t-2) \Rightarrow X_1(S) = \frac{e^{-2S}}{S}$$

$$Y(S) = X_1(S) \cdot X_2(S) = \frac{1}{S+3} \cdot \frac{e^{-2S}}{S} = \frac{e^{-2S}}{S(S+3)}.$$

Let $Y_1(S) = \frac{1}{S(S+3)} = \frac{A}{S} + \frac{B}{S+3}$

$1 = A(S+3) + BS$ Put $S = -3$

Put $S = 0 \Rightarrow A = 1/3$ $1 = -3B \Rightarrow B = -1/3.$

$$Y_1(S) = 1/3 \frac{1}{S} - \frac{1}{3} \frac{1}{S+3}$$

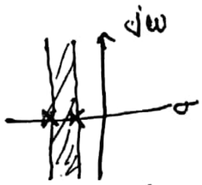
$$y_1(t) = \frac{1}{3}u(t) - \frac{1}{3}e^{-3t}u(t).$$

$e^{-2S} \cdot Y_1(S) \xleftrightarrow{L} y_1(t-2)$ ($\because$ By time shifting property).

$$\therefore y(t) = \frac{1}{3}u(t-2) - \frac{1}{3}e^{-3(t-2)}u(t-2)$$

Qn. Find the inverse LT of $X(S) = \frac{2}{(S+3)(S+2)}$ ROC: $-3 < Re(s) < -2$

Here the pole $S = -3$ lies to the left of ROC, hence the pole give rise to a causal signal.



The pole $S = -2$ lies to the right of ROC, hence the pole give rise to a non causal signal.

left $\Rightarrow -u(-t)$

right $\Rightarrow u(t)$

$$X(S) = \frac{2}{(S+3)(S+2)} = \frac{A}{S+3} + \frac{B}{S+2}$$

$2 = A(S+2) + B(S+3)$

Put $S = -3 \Rightarrow A = -2$ Put $S = -2 \Rightarrow B = 2$

$$\therefore X(S) = -2\underbrace{\frac{1}{S+3}}_{right} + 2\underbrace{\frac{1}{S+2}}_{\cancel{right}\ left}$$

Taking inv. $\Rightarrow x(t) = -2e^{-3t}u(t) - 2e^{-2t}u(-t)$

Qn. Find the signal whose bilateral transform is

$$X(s) = \frac{1}{(s+5)(s+1)}, \quad -5 < Re(s) < -1$$

$$X(s) = \frac{1}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$$

$1 = A(s+1) + B(s+5)$.

Put $s=-5 \Rightarrow A = -1/4$

Put $s=-1 \Rightarrow B = 1/4$.

$\therefore X(s) = -\frac{1}{4}\frac{1}{s+5} + \frac{1}{4}\frac{1}{s+1}$.

If ROC is $\underbrace{-5 < Re(s)}_{right} \overbrace{< -1}^{left}$

$$X(s) = \underbrace{-\frac{1}{4}\frac{1}{s+5}}_{right} + \underbrace{\frac{1}{4}\frac{1}{s+1}}_{left}$$

$$\therefore x(t) = -\frac{1}{4}e^{-5t}u(t) - \frac{1}{4}e^{-t}u(-t)$$

Qn. Determine the initial & final value of the function whose LT is given as $X(s) = \frac{5s+50}{s(s+5)}$.

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s \cdot \frac{5s+50}{s(s+5)} = \lim_{s\to\infty} \frac{s(5+\frac{50}{s})}{s(1+\frac{5}{s})}$$

$$= 5$$

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\frac{5s+50}{s(s+5)} = \frac{50}{5} = 10$$

Qn. Determine the inverse LT of $\frac{s+4}{2s^2+5s+3}$ using convolution method

$$X(s) = \frac{s+4}{2(s^2+\frac{5}{2}s+\frac{3}{2})} = \frac{s+4}{2(s+1)(s+3/2)} = \frac{A}{s+1} + \frac{B}{s+3/2}.$$

$A = 3$, $B = 10$

$\therefore X(s) = 3\frac{1}{s+1} + 10\frac{1}{s+3/2}$

$\therefore x(t) = 3e^{-t}u(t) + 10e^{-3/2 t}u(t)$

Laplace transform analysis of LTI Systems

consider a continuous time LTI s/m.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t).$$

$$y(t) = x(t) * h(t).$$

Taking LT

$$Y(s) = X(s) \cdot H(s)$$

$H(s) = \frac{Y(s)}{X(s)}$ is called the system function or transfer function of the s/m. It is the ratio of laplace transformed output to the laplace transformed input.

Relation between transfer function and differential eqn:

The n^{th} order LTI CT s/m described by the differential eqn is $\sum_{K=0}^{N} a_K \frac{d^K}{dt^K} y(t) = \sum_{K=0}^{M} b_K \frac{d^K}{dt^K} x(t)$

Taking LT. on both sides

$\frac{d^K}{dt^K} x(t) = s^K X(s)$ (initial conditions are neglected)

$$\sum_{K=0}^{N} a_K s^K Y(s) = \sum_{K=0}^{M} b_K s^K X(s)$$

$$Y(s) \sum_{K=0}^{N} a_K s^K = X(s) \sum_{K=0}^{M} b_K s^K$$

$$\frac{Y(s)}{X(s)} = \frac{\sum_{K=0}^{M} b_K s^K}{\sum_{K=0}^{N} a_K s^K} = \frac{b_0 + b_1 s + \cdots b_{M-1} s^{M-1} + b_M s^M}{a_0 + a_1 s + \cdots a_{N-1} s^{N-1} + a_N s^N}$$

where $\frac{Y(s)}{X(s)}$ is called transfer function.

* $H(s)$ plays a major role in finding response of system to different inputs.

Steps to find system response, $y(t)$:

1) First, we find the LT of input $x(t)$.
2) Find $Y(s) = H(s) \cdot X(s)$
3) Then we take inverse LT to get $y(t)$.

Properties of System using transfer fun. and ROC:

pole-zero & ROC of s/m TF. $H(s)$ provide following information.

a) frequency response

b) causality.

c) stability

a) frequency response: is obtained by replacing $s = j\omega$ in the TF. $H(s)$.

b) causality: If the ROC of LTI s/m must be entire region in the s-planes to the right of the right most pole, then that s/m is causal.

c) Stability:

* If all the poles of $H(s)$ must lie in the left half of s-plane, then the s/m is causal and stable.

* The System is marginally stable if poles of $H(s)$ are on the '$j\omega$' axis. * No repeated pole should be in the imaginary axis.

Problems:

Qn. The transfer fun. of LTI s/m is given by

$H(s) = \frac{2s-1}{s^2+3s+2}$. Determine the impulse response.

$$H(s) = \frac{2s-1}{s^2+3s+2} = \frac{2s-1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$A = 5$, $B = -3$.

$$\therefore H(s) = 5\frac{1}{s+2} - 3\frac{1}{s+1}.$$

Taking inv. LT

$$h(t) = 5e^{-2t}u(t) - 3e^{-t}u(t)$$

$\downarrow$

impulse response.

Qn. Determine the steady state response of the following s/m to unit step excitation. $H(s) = \frac{s+1}{s^2+3s+2}$.

i) i/p $x(t) = u(t)$.

$X(s) = \frac{1}{s}$.

ii) $Y(s) = H(s) \cdot X(s)$.

$$= \frac{s+1}{(s^2+3s+2)} \cdot \frac{1}{s} = \frac{\cancel{(s+1)}}{s\,\cancel{(s+1)}(s+2)}$$

$$Y(s) = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}.$$

$A = 1/2$, $B = -1/2$.

$\therefore Y(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$.

iii) Taking inv. LT.

$$y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t)$$ //

Qn. Determine the s/m response $y(t)$ for a system given below to an i/p $x(t) = e^{-3t}u(t)$ and $H(s) = \frac{2s^2+6s+6}{s^2+3s+2}$.

i) $x(t) = e^{-3t}u(t) \Rightarrow X(s) = \frac{1}{s+3}$

ii) $Y(s) = H(s) \cdot X(s)$

$$= \frac{2s^2+6s+6}{s^2+3s+2} \times \frac{1}{s+3} = \frac{(2s^2+6s+6)}{(s+1)(s+2)(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}.$$

$A = 1$, $B = -2$, $C = \frac{3}{2}$.

$\therefore Y(s) = \frac{1}{s+1} - 2\frac{1}{s+2} + \frac{3}{2}\frac{1}{s+3}$.

iii) Taking inv. LT

$$y(t) = e^{-t}u(t) - 2e^{-2t}u(t) + \frac{3}{2}e^{-3t}u(t)$$

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Qn. Check whether the following signals are causal or not

1) $h(t) = e^{-2t}u(t)$ 2) $h(t) = e^{-|t|}$

1) $h(t) = e^{-2t}u(t) \Rightarrow H(s) = \frac{1}{s+2}$ ROC : $\sigma > -2$

∵ The ROC is to the right of right most pole $s=-2$

Hence the system is causal.

2) $h(t) = e^{-|t|}$

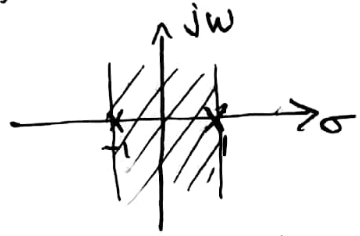
$$H(s) = \int_{-\infty}^{0} e^{t} e^{-st} dt + \int_{0}^{\infty} e^{-t} e^{-st} dt$$

$h(t) = e^{t}u(-t) + e^{-t}u(t)$

$\underbrace{\frac{-1}{s-1}}_{\sigma<1} + \underbrace{\frac{1}{s+1}}_{\sigma>-1}$

$$= -\frac{1}{s-1} + \frac{1}{s+1} = \frac{-(s+1)+s-1}{(s-1)(s+1)} = \frac{-s-1+s-1}{(s-1)(s+1)}$$

$$= \frac{-2}{(s-1)(s+1)}, \quad \text{ROC} : -1 < \sigma < 1$$



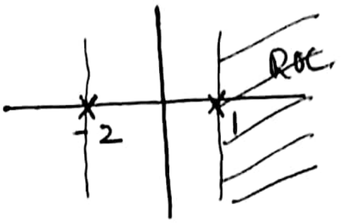
The right most pole is at $s=1$. The ROC is not to the right of the right most pole. Hence the system is not causal.

Qn. Test the causality and stability of the system $h(t) = 2e^{-2t}u(t) - e^{t}u(t)$

Ans $h(t) = 2e^{-2t}u(t) - e^{t}u(t)$.

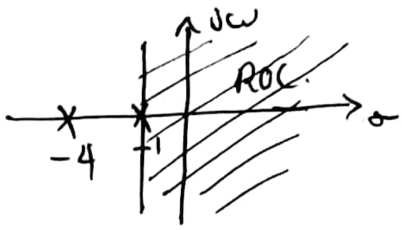
$$H(s) = 2\underbrace{\frac{1}{s+2}}_{ROC:\ \sigma > -2} - \underbrace{\frac{1}{s-1}}_{ROC:\ \sigma > 1} = \frac{2(s-1)-(s+2)}{(s+2)(s-1)}$$

$$= \frac{s-4}{(s+2)(s-1)}.$$



The right most pole is at $s=1$. The ROC is to the right of the right most pole. ∴ the s/m is causal. The pole $s=1$ which lies in right half of s-plane makes the s/m unstable.

Qn. Test the causality and stability of s/m whose s/m function is given as $H(s) = \frac{s-4}{(s+1)(s+4)}$. ROC: $\sigma > 1$;



The right most pole is at $s=-1$. The ROC is to the right of the right most pole. ∴ the s/m is causal. All the poles are in left half of s-plane. ∴ The system is stable.

Qn. Test whether the s/m $H(s) = \frac{s-4}{s^2(s+1)}$ is stable or not.

∴ There are two poles repeated at the origin.

∴ S/m is unstable.

Determining the frequency response from poles and zeros:

Step(1) from the poles - zeros, write the system function, $H(s)$

Step(2) Find $H(s)\big|_{s=j\omega}$, we get the freq. response.

Qn. Determine the frequency response of the system whose zero of $H(s)$ is $s=0.5$ and poles of $H(s)$ are $s=-2$ and $s=-1$

$$H(s) = \frac{s-0.5}{(s+2)(s+1)}$$

$$\text{freq. response},\ H(j\omega) = \frac{j\omega-0.5}{(j\omega+2)(j\omega+1)} = \frac{j\omega-0.5}{j^2\omega^2+3j\omega+2}$$

$$= \frac{j\omega-0.5}{-\omega^2+3j\omega+2}$$

Solution of differential eqns using Laplace transform:

Time differentiation property: with initial conditions

$$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s) - x(0)$$

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{L} s^2X(s) - sx(0) - \frac{dx(0)}{dt}$$

$$\frac{d^nx(t)}{dt^n} \xleftrightarrow{L} s^nX(s) - s^{n-1}x(0) \cdots - \frac{d^{n-1}}{dt^{n-1}}x(0).$$

without initial conditions:

$$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s)$$

$$\frac{d^2x(t)}{dt} \xleftrightarrow{L} s^2X(s)$$

$$\frac{d^nx(t)}{dt} \xleftrightarrow{L} s^nX(s).$$

Qn. By using LT, solve the following differential eqn:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}. \text{ if } y(0) = 2 \quad \frac{dy(0)}{dt} = 1$$

and $x(t) = e^{-t}u(t)$.

$$x(t) = e^{-t}u(t) \Longrightarrow X(s) = \frac{1}{s+1}$$

$$\frac{d^2y(t)}{dt} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

Taking LT

$$\left[s^2 Y(s) - sy(0) - \frac{d}{dt}y(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = sX(s) - x(0)$$

$$\left[s^2Y(s) - 2s - 1\right] + 3\left[sY(s) - 2\right] + 2Y(s) = sX(s).$$

$$s^2Y(s) - 2s - 1 + 3sY(s) - 6 + 2Y(s) = sX(s).$$

$$Y(s)\left[s^2 - 3s + 2\right] - 2s - 7 = sX(s).$$

$$Y(s)\left[s^2 - 3s + 2\right] = (2s+7) + sX(s)$$

$$Y(s) = \frac{(2s+7)sX(s)}{s^2 - 3s + 2} = \frac{s(2s+7)}{(s^2-3s+2)} + \frac{s}{(s+1)(s^2-3s+1)} \quad \left(\because X(s) = \frac{1}{s+1}\right)$$

$$Y(s) = \frac{(2s+7)(s+1) + s}{(s+1)(s^2-3s+1)} = \frac{2s^2+10s+7}{(s+1)(s+1)(s+2)}.$$

$$Y(s) = \frac{2s^2+10s+7}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)}.$$

$$2s^2+10s+7 = A(s+1)(s+2) + B(s+2) + C(s+1)^2.$$

$$A = 7 \qquad B = -1 \qquad C = -5.$$

$$\therefore Y(s) = 7\frac{1}{s+1} + -1\frac{1}{(s+1)^2} - 5\frac{1}{s+2},$$

Taking inverse LT

$$y(t) = 7e^{-t}u(t) - te^{-t}u(t) - 5e^{-2t}u(t)$$

Qn. Find the system transfer function of the following diff. eqn.
$$\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = 3\frac{d^2x(t)}{dt^2} + 7\frac{dx(t)}{dt} + 5x(t)$$

Taking LT (neglect initial conditions)

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = 3s^2X(s) + 7sX(s) + 5X(s).$$

$$Y(s)[s^3+6s^2+11s+6] = X(s)[3s^2+7s+5].$$

System function, $H(s) = \frac{Y(s)}{X(s)} = \frac{3s^2+7s+5}{s^3+6s^2+11s+6}.$

Qn. Find the impulse response and the step response of the system $H(s) = \frac{s+2}{s^2+5s+4}$

Impulse response:

$$H(s) = \frac{s+2}{s^2+5s+4} = \frac{(s+2)}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$A = 1/3$ and $B = 2/3$.

$$\therefore H(s) = \frac{1}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s+4}$$

Taking inv, $h(t) = \frac{1}{3}e^{-t}u(t) + \frac{2}{3}e^{-4t}u(t)$

Step response:

For step response, $x(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$

$$H(s) = \frac{Y(s)}{X(s)} \Rightarrow Y(s) = H(s).X(s)$$

$$= \frac{(s+2)}{(s+1)(s+4)}\cdot\frac{1}{s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$A = 1/2$, $B = -1/3$ and $C = -1/6$.

$$\therefore Y(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{3}\frac{1}{s+1} - \frac{1}{6}\frac{1}{s+4}$$

Taking inv., $y(t) = \frac{1}{2}u(t) - \frac{1}{3}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t).$

Qn. A system is described by the following differential eqn: $\frac{d^2y(t)}{dt} + 7\frac{dy(t)}{dt} + 12y(t) = x(t)$.
Determine the total response of the s/m to the i/p $x(t) = u(t)$. The initial conditions are $y(0) = -2$, $\frac{dy(0)}{dt} = 0$

Natural response (zero input response).

$$\frac{d^2y(t)}{dt} + 7\frac{dy(t)}{dt} + 12y(t) = x(t).$$

$$\cancel{\frac{d^2y(t)}{dt} + 7\frac{dy(t)}{dt}}$$

$$s^2Y(s) - sy(0) - \frac{d}{dt}y(0) + 7[sY(s) - y(0)] + 12Y(s) = X(s) \rightarrow ①$$

$$s^2Y(s) + 2s - 0 + 7sY(s) + 14 + 12Y(s) = 0$$

$$Y(s)[s^2+7s+12] + 14 + 2s = 14$$

$$Y(s) = \frac{-14-2s}{s^2+7s+12} = \frac{-14-2s}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$-2s - 14 = A(s+4) + B(s+3)$$

Put $s = -3 \Rightarrow -8 = A$, Put $s = -4 \Rightarrow -6 = -B \Rightarrow B = 6$

$$\therefore Y(s) = -8\frac{1}{s+3} + 14\frac{1}{s+4}$$

Taking in, $y(t) = -8e^{-3t}u(t) + 6e^{-4t}u(t) \rightarrow ②$

Forced response (zero state response).

$① \Rightarrow s^2Y(s) + 7sY(s) + 12Y(s) = X(s)$. | $x(t) = u(t)$, $X(s) = \frac{1}{s}$

$$Y(s)[s^2+7s+12] = 1/s.$$

$$Y(s) = \frac{1}{s(s^2+7s+12)} = \frac{1}{s(s+3)(s+4)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$1 = A(s+3)(s+4) + Bs(s+4) + Cs(s+3)$$

Put $s = 0 \Rightarrow 12A = 1 \Rightarrow A = 1/12$

Put $s = -3 \Rightarrow -3B = 1 \Rightarrow B = -1/3$

Put $s = -4 \Rightarrow 4C = 1 \Rightarrow C = 1/4$

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$$\therefore Y(S) = \frac{1}{12}\frac{1}{S} - \frac{1}{3}\frac{1}{S+3} + \frac{1}{4}\frac{1}{S+4}$$

Taking inverse, $y(t) = \frac{1}{12}u(t) - \frac{1}{3}e^{-3t}u(t) + \frac{1}{4}e^{-4t}u(t)$ $\rightarrow$ (3)

Total response = (2) + (3)

$$y(t) = -8e^{-3t}u(t) + 6e^{-4t}u(t) + \frac{1}{12}u(t) - \frac{1}{3}e^{-3t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

$$= \frac{1}{12}u(t) - \frac{25}{3}e^{-3t}u(t) + \frac{25}{4}e^{-4t}u(t)$$

$-14 - \frac{1}{3}$

$\frac{-42-1}{3} = \frac{-43}{3}$

$+6+\frac{1}{4}$

$\frac{32}{4}$

Qn. By using LT, solve the following differential eqn: $\frac{d^3y(t)}{dt^3} + 7\frac{d^2y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 12y(t) = x(t)$. If $\frac{dy(0)}{dt} = 0$, $\frac{d^2y(0)}{dt^2} = 0$, $y(0) = 0$ and $x(t) = \delta(t)$.

Taking LT

$$S^3Y(S) - S^2y(0) - S\frac{dy(0)}{dt} - \frac{d^2y(0)}{dt^2} + 7\left[S^2Y(S) - Sy(0) - \frac{dy(0)}{dt}\right] + 16\left[SY(S) - y(0)\right] + 12Y(S) = X(S).$$

Appling initial conditions,

$x(t) = \delta(t)$
$X(S) = 1$

$$S^3Y(S) + 7S^2Y(S) + 16SY(S) + 12Y(S) = 1$$

$$Y(S)\left[S^3 + 7S^2 + 16S + 12\right] = 1$$

$$\begin{array}{r|rrrr} -3 & 1 & 7 & 16 & 12 \\ & 0 & -3 & -12 & -12 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$S^3 + 7S^2 + 16S + 12$
$= (S+3)(S^2+4S+4)$
$= (S+3)(S+2)^2$

$$Y(S) = \frac{1}{S^3+7S^2+16S+12}$$

$$= \frac{1}{(S+3)(S+2)^2} = \frac{A}{S+3} + \frac{B}{S+2} + \frac{C}{(S+2)^2}$$

$A = 1$, $B = -1$ and C 1

$$\therefore Y(S) = \frac{1}{S+3} - \frac{1}{S+2} + \frac{1}{(S+2)^2}.$$

Taking inv

$$y(t) = e^{-3t}u(t) - e^{-2t}u(t) + te^{-2t}u(t).$$

Qn. Consider the RLC ckt shown in ~~given~~ fig. given below with $L = 1H$, $C = 1F$ and $R = 2.5\,\Omega$. Derive an expression for the o/p voltage $V_o(t)$ if the input is an unit step. Assume zero initial conditions.

$$L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt + R\,i(t) = V_i(t) \Rightarrow \frac{di(t)}{dt} + \int i(t)\,dt + 2.5\,i(t) = u(t) \rightarrow ①$$

$$V_o(t) = i(t).R. \Rightarrow V_o(t) = 2.5\,i(t) \rightarrow ②$$

Taking LT $\Rightarrow$ ① $\Rightarrow$ $S\,I(S) + \frac{I(S)}{S} + 2.5\,I(S) = \frac{1}{S} \Rightarrow I(S)\left[S + \frac{1}{S} + 2.5\right] = \frac{1}{S}$

$$I(S) = \frac{1}{S(S + \frac{1}{S} + 2.5)} \rightarrow ③$$

② $\Rightarrow$ $V_o(S) = 2.5\,I(S) \rightarrow ④$. Sub. $I(S)$ from ③ in ④

$$V_o(S) = \frac{2.5}{S^2 + 2.5S + 1} = \frac{2.5}{(S+2)(S+0.5)} = \frac{A}{S+2} + \frac{B}{S+0.5}$$

$2.5 = A(S+0.5) + B(S+2)$

Put $S = -2 \Rightarrow -1.5A = 2.5 \Rightarrow A = \frac{-2.5}{1.5} = \frac{-5}{3}$

Put $S = -0.5 \Rightarrow 1.5B = 2.5 \Rightarrow B = 5/3$.

$$\therefore V_o(S) = -\frac{5}{3}\frac{1}{S+2} + \frac{5}{3}\frac{1}{S+0.5}$$

Taking inv.

$$V_o(t) = -\frac{5}{3}e^{-2t}u(t) + \frac{5}{3}e^{-0.5t}u(t)$$

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Sampling:

* Sampling is the process of converting continuous time signal to discrete time signal ie discretization of continuous time signal.

Sampling theorem:

A band limited signal $x(t)$ is reconstructed from its samples $x(nT)$, if the sampling frequency f_s must be $\geq 2f_m$ ie sampling frequency must be atleast twice the highest frequency present in the signal.

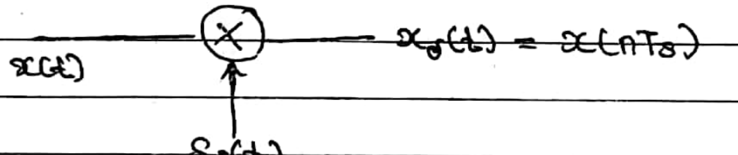
Nyquist rate or Nyquist frequency:

* two times max. frequency present in the signal ($2f_m$).

* The minimum rate at which a signal can be sampled and still be reconstructed from its samples is called Nyquist rate.

* Nyquist interval $= 1/\text{Nyquist frequency}$.

Derivation:



(impulse train) (Dirac comb function)
(Sampling function)

Sampling function : $\delta_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

Fourier transform of sampling function is also a sampling function

ie $\delta_s(t) \xleftrightarrow{FT} \delta_s(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$

from the figure:

$$x_\delta(t) = x(t) \times \delta_\delta(t)$$

Taking FT

$$X_\delta(\omega) = \frac{1}{2\pi}\left[X(\omega) * \delta_\delta(\omega)\right]$$

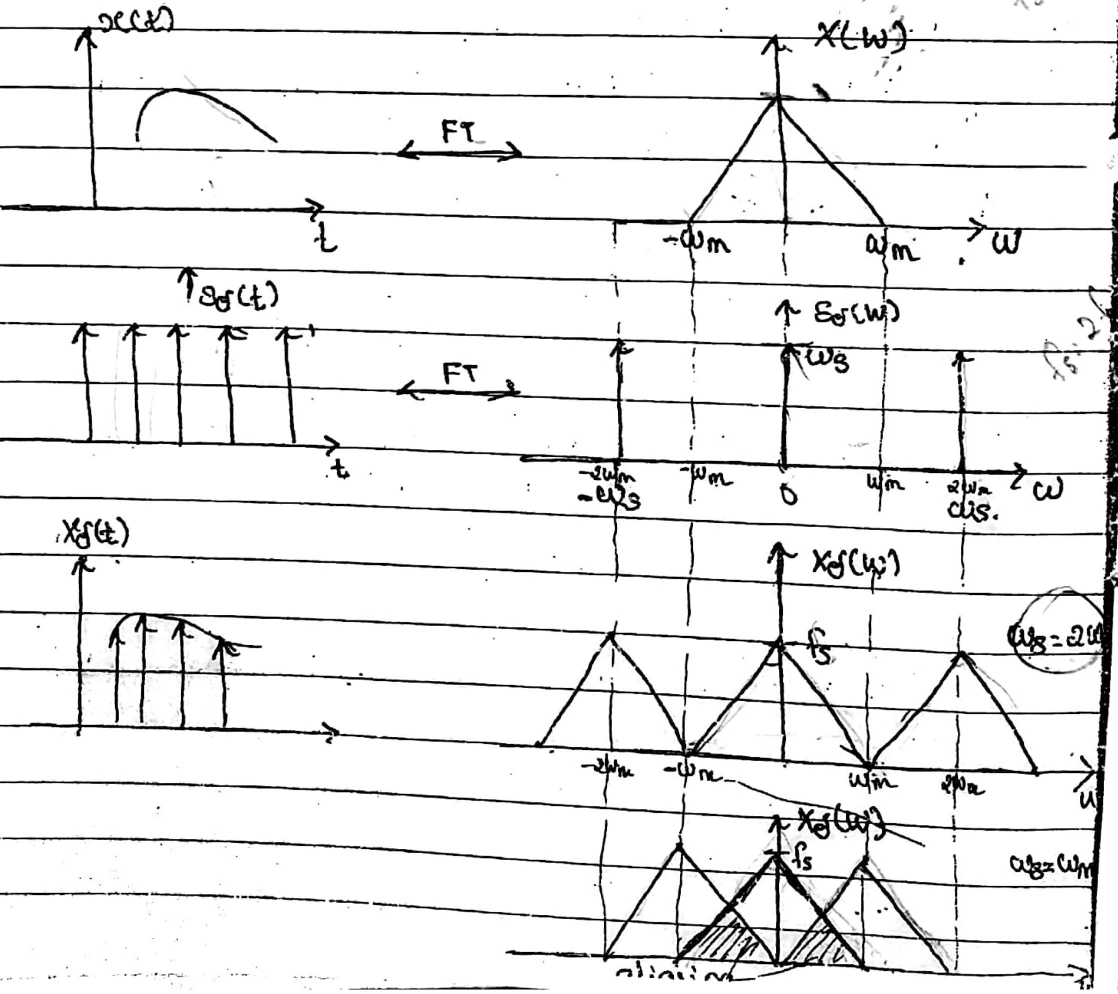
$$X_\delta(\omega) = \frac{1}{2\pi}\left[X(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

$$= \frac{\omega_s}{2\pi}\left[X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

$x(n) * \delta(n-k) = x(n-k)$

$$= \frac{1}{2\pi} 2\pi f_s \left[X(\omega - n\omega_s)\right]$$

$$\boxed{X_\delta(\omega) = f_s \cdot X(\omega - n\omega_s)}$$



If sampling frequency less than Nyquist frequency, ie $f_s < 2f_m$, then the high frequency interfere with the low frequency. This overlapping is called aliasing.

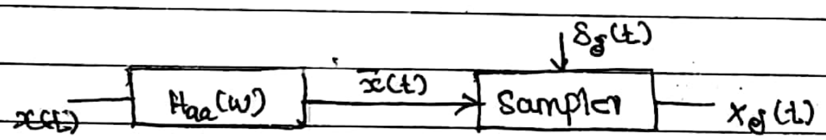
The effects of aliasing are:-

* Distortion in signal recovery is generated when the high and low frequencies interfere with each other.
* The data is lost and it cannot be recovered.

Different methods are available to avoid aliasing

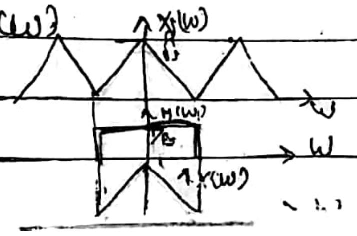
* To increase the sampling frequency f_s so that $f_s > 2f_m$.
* To put anti-aliasing filter before the signal $x(t)$ is sampled.

Anti-aliasing filter:



The anti-aliasing filter $H_{aa}(\omega)$ put before the sampler is shown in figure. $x(t)$ is the continuous time signal and is passed through the anti-aliasing filter $H_{aa}(\omega)$, ~~which gives the output $\bar{x}(t)$. The continuous signal $x(t)$ is passed~~ whose cut off frequency is $f_s/2$ and gives the output $\bar{x}(t)$. The anti-aliasing filter eliminates all the frequency components of $x(t)$ beyond $f_s/2$ before sampling.

Signal Reconstruction: Sampled signal $X_s(\omega)$ is passed through a low pass filter, $H(\omega)$, we get the signal $X(\omega)$. The original signal $x(t)$ is obtained by taking inverse of $X(\omega)$.



Q. Find the Nyquist rate and Nyquist interval of the following signals. a) $x(t) = \sin 200\pi t$

$$\omega_m = 200\pi$$
$$2\pi f_m = 200\pi$$
$$f_m = 200\pi / 2\pi$$
$$= 100\ Hz$$

Nyquist freq. $= 2f_m = 2\times 100 = 200\ Hz$

Nyquist interval $= 1/\text{Nyq. freq} = 1/200 = 0.5\times 10^{-2}$ secs.

b) $x(t) = 2 + 3\cos 100\pi t + 2\sin 200\pi t$

$$\omega_m = 200\pi$$
$$2\pi f_m = 200\pi$$
$$f_m = 100\ Hz$$

Nyq. freq $= 2f_m = 200\ Hz$.

Nyq. interval $= 1/200$ secs.

c) $x(t) = \dfrac{\sin 100\pi t}{\pi t}$

$$\omega_m = 100\pi$$

Nyq. freq, $2f_m = 2\times 50 = 100\ Hz$.

Nyq. interval $= 1/100$ secs.

d) $x(t) = \sin^2 200\pi t$

$$= \frac{1 - \cos 400\pi t}{2}$$

$\omega_m = 400\pi \Rightarrow 2\pi f_m = 400\pi$

Nyq. freq $= 400\ Hz$ $\qquad f_m = 200\ Hz$

Nyq. interval $= 1/400$ secs.

MODULE 4 & 5

Introduction:

Transform techniques are an important tool in the analysis of signals and systems.

Fourier series: (CTFS and DTFS)
→ For the analysis of periodic signals.

Fourier transform: (DTFT and CTFT).
→ For the analysis of aperiodic signals.

Simple and systematic transforms:

Laplace transform:
→ For the analysis of continuous time signals and systems.

Z-transform:
→ For the analysis of discrete time signals and systems.

* The laplace transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the begining of the process itself. To solve the differential eqns which are in time domain, they are first converted into algebraic eqns in frequency domain using laplace transform, the algebraic eqns. are manipulated in s-domain and the result obtained in freq. domain is converted back into time domain using inverse laplace transform.

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Z- transform:

Z- transform has the advantage that it is a simple and systematic method and the complete soln. can be obtained in one step and the initial conditions can be introduced in the begining of the process itself. To solve difference eqns which are in time domain they are converted first into algebraic eqns. in z-d using z-transform, the algebraic eqns. are manipulated in z-domain and the result obtained is converted back into time domain using inverse z-transform.

The bilateral or two sided z-transform of a discrete time signal $x(n)$ is defined as:

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}.$$

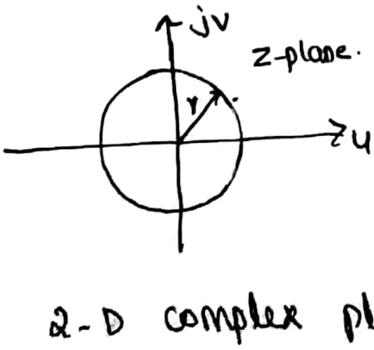
The unilateral or one sided z-transform of $x(n)$ is defined as $X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$.

where Z is a complex variable.

$$Z = \underset{\substack{\downarrow \\ \text{real part} \\ \text{of } Z}}{u} + j\underset{\substack{\downarrow \\ \text{imag. part} \\ \text{of } Z}}{v} = re^{j\Omega}$$

magnitude of Z, $r = \sqrt{u^2+v^2}$

phase of Z, $\Omega = \tan^{-1}(v/u)$



A 2-D complex plane with values of u on horizontal axis and the values of v on vertical axis as shown in figure is called Z-plane.

* The set of z values for which the summation converges is called region of convergence (ROC) for the transform.

Region of convergence:

Since z-transform is an infinite power series, it exists only for those values of z for which the series converges. The ROC of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

Z-transform and ROC of finite duration sequence

Right sided sequence:

Consider the sequence $x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$ ($x(0)$, $x(1)$, $x(2)$, $x(3)$)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{3} x(n) z^{-n}.$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

In the above summation when $z=0$, all the terms except the first term become infinite. ie $X(z)$ converges for all values of z except at $z=0$

∴ The ROC for finite duration right sided signal is entire z-plane except at $z=0$.
Mathematically ROC: $|z| > 0$

Left sided sequence:

Consider the sequence $x(n) = \{1, 2, 1, \underset{\uparrow}{3}\}$ ($x(-3)$, $x(-2)$, $x(-1)$, $x(0)$)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-3}^{0} x(n) z^{-n}$$

$$X(z) = x(-3)z^3 + x(-2)z^2 + x(1)z^1 + x(0)$$
$$= z^3 + 2z^2 + z + 3$$

In the above summation when $z = \infty$, all the terms except the last term become infinite ie $X(z)$ converges for all values of z except at $z = \infty$

∴ The ROC for finite duration left sided signal is entire z-plane except at $z = \infty$
mathematically ROC: $|z| < \infty$

Double sided sequence: (Two sided sequence)

A signal that has finite duration on both left and right sides is known as double sided s/l. In this case the ROC is the entire z-plane except at $z = 0$ and $z = \infty$.

Consider the sequence $x(n) = (\overset{x(-2)}{2}, \overset{x(-1)}{1}, \underset{\uparrow}{\overset{x(0)}{1}}, \overset{x(1)}{2})$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^{1} x(n) z^{-n}$$

$$= x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1}$$

$$= 2z^2 + z + 1 + 2z^{-1}$$

The above expression for $X(z)$ becomes infinity at $z = 0$ and $z = \infty$. Hence the ROC is the entire z-plane except at $z = 0$ and $z = \infty$

This is explained mathematically by writing the ROC as, ROC: $0 < |z| < \infty$.

Z-transform and ROC of infinite duration sequence:

Right sided (positive time exponential) sequence:

A right sided infinite sequence is defined as

$$x(n) = a^n \quad n \geq 0$$

ie. $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

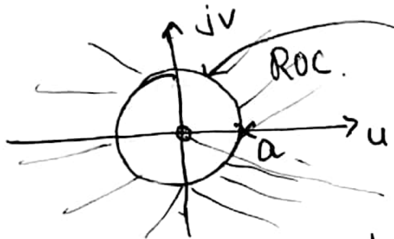
$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$X(z)$ converges if $|az^{-1}| < 1$

$$\frac{|a|}{|z|} < 1 \Rightarrow |z| > |a|$$

ie $X(z)$ converges for all points external to the circle of radius 'a' in ~~z-plane~~. ∴ the ROC of $X(z)$ is exterior of the circle of radius 'a' in ~~z-plane~~ as shown in fig. given below.



*

Values of z for which $X(z) = 0$ is called zeros of $X(z)$ while the values of z for which $X(z) = \infty$ is called poles of $X(z)$. poles are usually indicated by 'x' at $z = a$ and zeros by 'o' at $z = 0$

Left sided (negative time exponential) sequence:

A left sided infinite sequence is defined as

$$x(n) = -b^n u(-n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

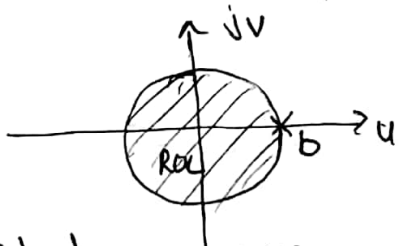
$$= \sum_{n=-\infty}^{\infty} -b^n u(-n) z^{-n} = -\sum_{n=-\infty}^{0} b^n z^{-n} = -\sum_{n=\infty}^{0} b^{-n} z^{n}.$$

$$= -\sum_{n=0}^{\infty} (b^{-1}z)^n = -\frac{1}{1-b^{-1}z} = \frac{b}{z-b}$$

$x(z)$ converges $|b^{-1}z| < 1$

$$\frac{|z|}{|b|} < 1 \Rightarrow |z| < |b|$$

ie $x(z)$ converges for all point internal to the circle of radius 'b'. ∴ The ROC of $x(z)$ is the interior of the circle of radius 'b' ~~in~~ ~~z~~- ~~plane~~.



Double sided sequence:

$$x(n) = a^n u(n) - b^n u(-n) \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} - \sum_{n=-\infty}^{\infty} b^n u(-n) z^{-n}$$

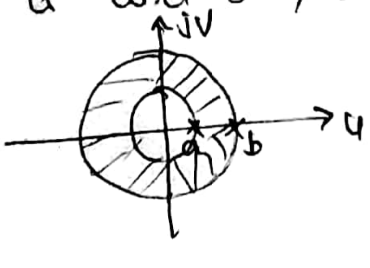
$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=-\infty}^{0} b^n z^{-1}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=0}^{\infty} (b^{-1}z)^n$$

$$= \frac{1}{1-az^{-1}} - \frac{1}{1-b^{-1}z} =$$

The 1st term converges if $|az^{-1}| < 1 \Rightarrow |z| > |a|$
The 2nd term converges if $|b^{-1}z| < 1 \Rightarrow |z| < |b|$

∴ ROC is the region between two circles of radius 'a' and 'b', when $|b| > |a|$



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Summary:

Sequence	ROC.
1) finite right sided	entire z-plane except at $z=0$
2) finite left sided	entire z-plane except at $z=\infty$
3) finite double sided	entire z-plane ~~expt~~ except at $z=0$ & $z=\infty$
4) infinite right sided	exterior of the circle of radius 'a', $\|z\| > \|a\|$.
5) Infinite left sided	interior of the circle of radius 'b', $\|z\| < \|b\|$
6) Infinite double sided.	Region b/w the two circles of radius 'a' & 'b', where $\|b\| > \|a\|$, and $\|a\| < \|z\| < \|b\|$

Properties of ROC:

1) The ROC is a ring in the z-plane centered at the origin.
2) The ROC cannot contain any poles.
3) If $x(n)$ is a finite right sided sequence, then the ROC is the entire z-plane except at $z=0$.
4) If $x(n)$ is a finite left sided sequence, then the ROC is the entire z-plane except at $z=\infty$
5) If $x(n)$ is a finite double sided sequence, then the ROC is the entire z-plane except at $z=0$ and $z=\infty$
6) If $x(n)$ is an infinite double sided sequence, then the ROC will consist of a ring in the z-plane bounded on the interior and exterior by a pole.
7) The ROC of a stable system contains the unit circle.
8) The ROC must be a connected region.

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Relation between z-transform and FT

z-transform of $x(n)$ is $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Put $z = e^{j\Omega}$

$\therefore X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} =$ Fourier transform of $x(n)$.

Problems:

Qn) Find the z-transform & ROC of the following signals

a) $x(n) = u(n)$

b) $x(n) = \delta(n)$

c) $x(n) = e^{j\Omega_0 n} u(n)$

d) $x(n) = u(-n)$

e) $x(n) = a^{|n|}$ for $|a|<1$ ~~and~~ ~~|a|>1~~

f) $x(n) = a^{-n} u(n-1)$

g) $x(n) = u(n) - u(n-6)$

h) $x(n) = (\underset{\uparrow}{2}, -1, 0, 3, 4)$

i) $x(n) = (1, -2, 3, -1, \underset{\uparrow}{2})$

j) $x(n) = (5, 3, \underset{\uparrow}{-2}, 0, 4, -3)$

Ans:

a) $x(n) = u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} u(n)\, z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

$X(z)$ converges if $|z^{-1}| < 1 \Rightarrow \frac{1}{|z|} < 1 \Rightarrow |z| > 1$

ROC: $|z| > 1$.

b) $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n} = z^{-n}\Big|_{n=0}$$

$= 1$ ROC: entire z plane.

c) $x(n) = e^{j\Omega_0 n} u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{j\Omega_0 n} z^{-n} = \sum_{n=0}^{\infty} (e^{j\Omega_0} z^{-1})^{n} = \frac{1}{1-e^{j\Omega_0} z^{-1}}$$

$$= \frac{z}{z-e^{j\Omega_0}}$$

$X(z)$ converges if $|e^{j\Omega_0} z^{-1}| < 1 \Rightarrow \frac{|e^{j\Omega_0}|}{|z|} < 1$

$|z| > |e^{j\Omega_0}| \Rightarrow |z| > 1$

ROC: $|z| > 1$

$e^{j\Omega_0} = \cos\Omega_0 + j\sin\Omega_0$

$|e^{j\Omega_0}| = \sqrt{\cos^2\Omega_0 + \sin^2\Omega_0}$

$= \sqrt{1} = 1$

d) $x(n) = u(-n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} u(-n)\, z^{-n} = \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{n=\infty}^{0} z^{n} = \sum_{n=0}^{\infty} z^{n} = \frac{1}{1-z}$$

$X(z)$ converges if $|z| < 1$

$\therefore$ ROC: $|z| < 1$

e) $x(n) = a^{|n|}$ $\qquad |n| = \begin{cases} n & \text{for } n \geq 0 \\ -n & \text{for } n < 0 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^{n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (a^{-n} z^{-n}) + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

$$= \sum_{n=\infty}^{1} a^{n} z^{n} + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

$$= \sum_{n=1}^{\infty} (az)^{n} + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

$$= \left(\sum_{n=0}^{\infty} (az)^{n} - 1 \right) + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

$$= \left(\frac{1}{1-az} - 1 \right) + \frac{1}{1-az^{-1}}$$

$$= \frac{1-(1-az)}{1-az} + \frac{z}{z-a}$$

$$= \frac{az}{1-az} + \frac{z}{z-a} = \frac{az(z-a) + z(1-az)}{(1-az)(z-a)}$$

$$= \frac{z(1-a)}{(1-az)(z-a)}$$

$$\rightarrow \frac{-z}{z-\frac{1}{a}} + \frac{z}{z-a}$$

$$= \frac{z(z-\frac{1}{a}) - z(z-a)}{(z-\frac{1}{a})(z-a)}$$

$$= \frac{z^2 - \frac{z}{a} - z^2 + za}{(z-\frac{1}{a})(z-a)} = \frac{za - \frac{z}{a}}{(z-\frac{1}{a})(z-a)}$$

ROC: * $|az| < 1$

$|a||z| < 1$

$|z| < \frac{1}{|a|}$

* $|az^{-1}| < 1$

$\frac{|a|}{|z|} < 1$

$|z| > |a|$

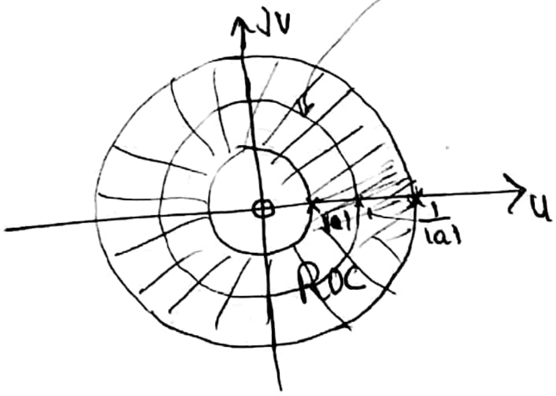
ROC: $|a| < |z| < \frac{1}{|a|}$

unit circle

zeros: $z = 0$

poles: $az = 1$ and $z = a$

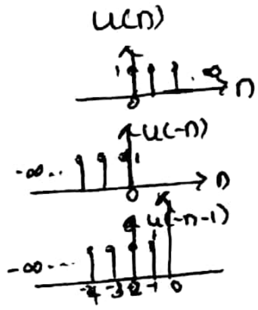
$z = 1/a$



f) $x(n) = a^{-n} u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^{-n} u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} = \sum_{n=\infty}^{1} a^{n} z^{n}$$



$$= \sum_{n=1}^{\infty} a^{n} z^{n} = \sum_{n=0}^{\infty} (az)^{n} - 1$$

$$= \frac{1}{1-az} - 1 = \frac{1-(1-az)}{1-az}$$

$$= \frac{az}{1-az} = \frac{-z}{z-\frac{1}{a}}$$

$X(z)$ converges $|az| < 1$

ROC: $|z| < \frac{1}{|a|}$

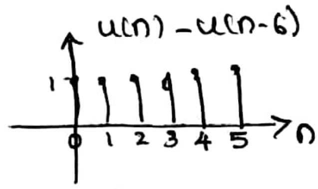
g) $x(n) = u(n) - u(n-6)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [u(n) - u(n-6)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(n) z^{-n} - \sum_{n=-\infty}^{\infty} u(n-6) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} - \sum_{n=6}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{5} z^{-n} = \underbrace{1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}}_{\text{finite right sided seq.}}$$



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{5} x(n) z^{-n}$$

$$= \sum_{n=0}^{5} z^{-n}$$

$\therefore$ ROC: Entire z-plane except at $z = 0$

Qn. Find z-transform of $x(n) = \cos\Omega n \, u(n)$

$$= \left[\frac{e^{j\Omega n}}{2} + \frac{e^{-j\Omega n}}{2} \right] u(n)$$

W.K.T $Z\{e^{j\Omega n} u(n)\} = \dfrac{1}{1 - e^{j\Omega} z^{-1}}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[\frac{e^{j\Omega n}}{2} + \frac{e^{-j\Omega n}}{2} \right] u(n) z^{-n}$$

$$= \frac{1}{2} \underbrace{\sum_{n=-\infty}^{\infty} e^{j\Omega n} u(n) z^{-n}}_{\frac{1}{1 - e^{j\Omega} z^{-1}}} + \frac{1}{2} \underbrace{\sum_{n=-\infty}^{\infty} e^{-j\Omega n} u(n) z^{-n}}_{\frac{1}{1 - e^{-j\Omega} z^{-1}}}$$

$$\therefore X(z) = \frac{1}{2} \left[\frac{1}{1 - e^{j\Omega} z^{-1}} + \frac{1}{1 - e^{-j\Omega} z^{-1}} \right]$$

$$= \frac{1}{2} \left[\frac{z}{z - e^{j\Omega}} + \frac{z}{z - e^{-j\Omega}} \right]$$

$$= \frac{1}{2} \left[\frac{z(z - e^{-j\Omega}) + z(z - e^{j\Omega})}{(z - e^{j\Omega})(z - e^{-j\Omega})} \right]$$

$$\cancel{= \frac{1}{2} \left[\frac{z^2 - z e^{-j\Omega} + z^2 - z e^{j\Omega}}{z^2 - z e^{-j\Omega} - z e^{j\Omega} + 1} \right]}$$

$$\cancel{= \left[2z^2 - \right.} = \frac{1}{2} \left[\frac{z(z - e^{-j\Omega} + z - e^{j\Omega})}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1} \right]$$

$$= \frac{1}{2} \left[\frac{z(2z - (e^{j\Omega} + e^{-j\Omega}))}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1} \right]$$

$$= \frac{1}{2} \left[\frac{z(2z - 2\cos\Omega)}{z^2 - 2z\cos\Omega + 1} \right] = \frac{z(z - \cos\Omega)}{z^2 - 2z\cos\Omega + 1}$$

ROC: $|z| > 1$

Qn. Find the z-transform of $x(n) = \sin \Omega n \, u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \frac{1}{2j}\sum_{n=-\infty}^{\infty}\left[e^{j\Omega n} - e^{-j\Omega n}\right] u(n) z^{-n}$$

$$= \frac{1}{2j}\left\{ \underbrace{\sum_{n=-\infty}^{\infty} e^{j\Omega n} u(n) z^{-n}}_{\frac{z}{z-e^{j\Omega}}} - \underbrace{\sum_{n=-\infty}^{\infty} e^{-j\Omega n} u(n) z^{-n}}_{\frac{z}{z-e^{-j\Omega}}} \right\}$$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{j\Omega}} - \frac{z}{z-e^{-j\Omega}}\right] = \frac{1}{2j}\left[\frac{z(z-e^{-j\Omega}) - z(z-e^{j\Omega})}{(z-e^{j\Omega})(z-e^{-j\Omega})}\right]$$

$$= \frac{1}{2j}\left[\frac{z(\cancel{z}-e^{-j\Omega}-\cancel{z}+e^{j\Omega})}{z^2 - ze^{-j\Omega} - ze^{j\Omega}+1}\right]$$

$$= \frac{1}{2j}\left[\frac{z(e^{j\Omega}-e^{-j\Omega})}{z^2 - z(e^{j\Omega}+e^{-j\Omega})+1}\right]$$

$$= \frac{1}{\cancel{2j}}\left[\frac{z \cdot \cancel{2j}\sin\Omega}{z^2 - 2z\cos\Omega + 1}\right] = \frac{z\sin\Omega}{z^2-2z\cos\Omega+1}$$

ROC: $|z| > 1$

Qn. Find the z transform and ROC of $X(z)$ for $x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(-\frac{1}{3}\right)^n u(n)$. Also find the pole-zero location.

W.K.T $a^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$

$\therefore \left(\frac{5}{7}\right)^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{5}{7}z^{-1}}$

$\left(-\frac{1}{3}\right)^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1--\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$

$$\therefore X(z) = 3 \cdot \underbrace{\frac{1}{1-\frac{5}{7}z^{-1}}}_{X_1(z)} + 2\underbrace{\frac{1}{1+\frac{1}{3}z^{-1}}}_{X_2(z)}$$

$1 - \frac{10}{7}$ $\quad \frac{7-10}{7} = \frac{-3}{7}$

$$= 3\frac{z}{z-\frac{5}{7}} + 2\frac{z}{z+\frac{1}{3}}$$

$$= \frac{3z\left(z+\frac{1}{3}\right) + 2z\left(z-\frac{5}{7}\right)}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)} = \frac{3z^2 + z + 2z^2 - \frac{10}{7}z}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)}$$

$$= \frac{5z^2 - \frac{3}{7}z}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)} = \frac{z\left(5z-\frac{3}{7}\right)}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)}$$

$X_1(z)$ converges if $\left|\frac{5}{7}z^{-1}\right| < 1 \Rightarrow |z| > \frac{5}{7}$, $|z| > 0.71$

$X_2(z)$ converges if $\left|-\frac{1}{3}z^{-1}\right| < 1 \Rightarrow |z| > \frac{1}{3}$, $|z| > 0.33$

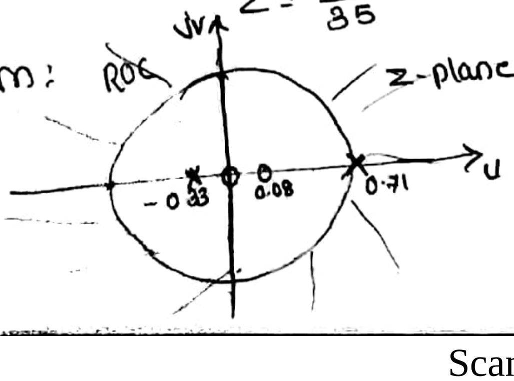
$\therefore X(z)$ converges if $|z| > 0.71$ $\quad \therefore$ ROC: $|z| > 0.71$

poles: $z = \frac{5}{7} = \underline{0.71}$ and $z = -\frac{1}{3} = \underline{-0.33}$.

zeros: $z = 0$ and $5z = \frac{3}{7}$

$z = \frac{3}{35} = 0.08$

Pole zero diagram:



Properties of z-transform:

1) **Linearity:** If $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

Then $x(n) = a x_1(n) + b x_2(n) \xleftrightarrow{z} X(z) = a X_1(z) + b X_2(z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$

$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$

$= a X_1(z) + b X_2(z)$.

2. **Time shifting:** If $x_1(n) \xleftrightarrow{z} X_1(z)$

Then $x(n) = x_1(n-k) \xleftrightarrow{z} X(z) = z^{-k} X_1(z)$.

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(n-k) z^{-n}$

Put $m = n-k$ $\Rightarrow n = m+k$ $\therefore X(z) = \sum_{m=-\infty}^{\infty} x_1(m) z^{-(m+k)} = \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} z^{-k}$

$= z^{-k} \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} = z^{-k} X_1(z)$.

* If $x(n)$ is one sided seq:

i) $x(n-m) \xleftrightarrow{z} z^{-m}\left[X(z) + \sum_{k=1}^{m} x(-k) z^{k}\right]$

ii) $x(n+m) \xleftrightarrow{z} z^{m}\left[X(z) - \sum_{k=0}^{m-1} x(k) z^{-k}\right]$

3. **Time Reversal:** If $x_1(n) \xleftrightarrow{z} X_1(z)$

Then $x(n) = x_1(-n) \xleftrightarrow{z} X(z) = X_1(z^{-1})$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(-n) z^{-n}$

Put $m = -n$ $\therefore X(z) = \sum_{m=\infty}^{-\infty} x_1(m) z^{m} = \sum_{m=-\infty}^{\infty} x_1(m) (z^{-1})^{-m}$

$= X_1(z^{-1})$.

4. **Multiplication by an exponential seq:**

If $x_1(n) \xleftrightarrow{z} X_1(z)$ then $x(n) = a^n x_1(n) \xleftrightarrow{z} X(z) = X_1(a^{-1}z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^n x_1(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} x_1(n) (a^{-1}z)^{-n} = X_1(a^{-1}z)$.

5. **Multiplication by n (ramp):** If $x_1(n) \xleftrightarrow{z} X_1(z)$

Then $x(n) = n x_1(n) \xleftrightarrow{z} X(z) = -z \frac{d}{dz} X_1(z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} n x_1(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(n)\, n \cdot z^{1} z^{-n-1} = z \sum_{n=-\infty}^{\infty} -x_1(n)(-n z^{-n-1})$

$= -z \sum_{n=-\infty}^{\infty} x_1(n) \frac{d}{dz} z^{-n}$.

$= -z \frac{d}{dz} \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = -z \frac{d}{dz} X_1(z)$

(OR) $X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$ ~~$\sum_{n=-\infty}^{\infty} n x_1(n) z^{-n}$~~.

$$\frac{dX_1(z)}{dz} = \sum_{n=-\infty}^{\infty} [x_1(n)] \frac{d}{dz} z^{-n} \Rightarrow \frac{dX_1(z)}{dz} = \sum_{n=-\infty}^{\infty} x_1(n) (-n) z^{-n-1}$$

$$= \frac{dX_1(z)}{dz} = -\sum_{n=-\infty}^{\infty} n x_1(n) z^{-1} z^{-n} = -z^{-1} \sum_{n=-\infty}^{\infty} [n x_1(n)] z^{-n}$$

$$-z \frac{dX_1(z)}{dz} = \sum_{n=-\infty}^{\infty} [n x_1(n)] z^{-n}$$

$$\therefore \; n x_1(n) \xleftrightarrow{z} -z \frac{dX_1(z)}{dz}.$$

6. Convolution: If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$

then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X(z) = X_1(z) \cdot X_2(z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] z^{-n}$.

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] z^{-n},$$ changing the order of summation

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}.$$

Put $m = n-k \Rightarrow n = m+k$.

$$\therefore X(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m) z^{-(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m) z^{-m} z^{-k} = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{m=-\infty}^{\infty} x_2(m) z^{-m}.$$

$$= X_1(z) X_2(z)$$

7. Initial value theorem: If $x(n) \xleftrightarrow{z} X(z)$

then $x(0) = \lim_{z\to\infty} X(z)$.

Proof: $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

As $z \to \infty$, all the terms vanish except $x(0)$.

$\therefore \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x(n) z^{-n} = x(0)$ ie $x(0) = \lim_{z\to\infty} X(z)$.

8. Final value theorem: If $x(n) \xleftrightarrow{z} X(z)$, then

$$x(\infty) = \lim_{z\to 1} (z-1) X(z).$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

Form the seq. $x(n+1) - x(n)$ and take its z-transform

$$Z\{x(n+1)-x(n)\} = \sum_{n=0}^{\infty}[x(n+1)-x(n)]Z^{-n}$$

$$= [x(1)-x(0)]Z^{-0} + [x(2)-x(1)]Z^{-1} + [x(3)-x(2)]Z^{-2}$$

$$+ \cdots \cancel{x(\infty)} \quad + [x(\infty)-x(\infty-1)] Z^{-\infty-1}$$

$$\longrightarrow (1)$$

By using shifting property.

$$x(n+m) \overset{Z}{\longleftrightarrow} Z^{m}\left[X(Z) - \sum_{K=0}^{m-1} x(K) Z^{-K}\right]$$

$$x(n+1) \overset{Z}{\longleftrightarrow} Z[X(Z) - \cancel{1}\, x(0)] = ZX(Z) - Zx(0)$$

$$Z\{x(n+1)-x(n)\} = ZX(Z) - Zx(0) - X(Z)$$

$$= (Z-1)X(Z) - ZX(0) \longrightarrow (2)$$

$$\therefore (Z-1)X(Z) - ZX(0) = [x(1)-x(0)] + [x(2)-x(1)]Z^{-1}$$

$$+ \cdots [x(\infty) - x(\infty-1)] Z^{-\infty}$$

Taking limit $Z \to 1$ on both sides.

$$\underset{Z\to 1}{Lt}\,(Z-1)X(Z) - ZX(0) = \underset{Z\to 1}{Lt}\left\{[x(1)-x(0)] + [x(2)-x(1)]Z^{-1} + \cdots [x(\infty)-x(\infty-1)]Z^{-\infty}\right.$$

$$\underset{Z\to 1}{Lt}\,(Z-1)X(Z) - \cancel{x(0)} = \cancel{x(1)} - x(0) + \cancel{x(2)} - \cancel{x(1)} + x(3) - \cancel{x(2)}$$

$$+ \cdots \quad x(\infty) - x(\infty-1).$$

$$= x(\infty) - \cancel{x(0)}.$$

$$x(\infty) = \underset{Z\to 1}{Lt}\,(Z-1)X(Z)$$

Time shifting (Time delay) property:

If $x(n)$ is one sided sequence

i) $x(n-m) \overset{Z}{\longleftrightarrow} z^{-m}\left[X(z) + \sum_{k=1}^{m} x(-k)\, z^{k}\right]$

ii) $x(n+m) \overset{Z}{\longleftrightarrow} z^{m}\left[X(z) - \sum_{k=0}^{m-1} x(k)\, z^{-k}\right]$

Proof:

$$Z\text{-transform of } x(n-m) = \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m)\, z^{m} z^{-m} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-m} z^{-(n-m)}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-(n-m)}$$

Put $l = n-m \Rightarrow$

$$\therefore \; = z^{-m} \sum_{l=-\infty}^{\infty} x(l)\, z^{-l}$$

$$= z^{-m}\left[\sum_{l=0}^{\infty} x(l)\, z^{-l} + \sum_{l=-m}^{-1} x(l)\, z^{-l}\right]$$

~~Put $l = -k$~~

$\therefore \cancel{= z^{-m}\left[\sum_{k=0}\right.}$

$$= z^{-m}\left[X(z) + \sum_{l=-m}^{-1} x(l)\, z^{-l}\right]$$

Put $l = -k$

$$\therefore \; = z^{-m}\left[X(z) + \sum_{k=m}^{1} x(-k)\, z^{k}\right]$$

$$= z^{-m}\left[X(z) + \sum_{k=1}^{m} x(-k)\, z^{k}\right]$$

$$x(n+m) \overset{z}{\longleftrightarrow} z^{m}\left[X(z) - \sum_{k=0}^{m-1} x(k)\, z^{-k}\right]$$

$$x(n+m) \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} x(n+m)\, z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n+m)\, z^{-(n+m)} \cdot z^{m}$$

$$= z^{m} \sum_{n=-\infty}^{\infty} x(n+m)\, z^{-(n+m)}$$

Put $l = n+m$

$$\therefore = z^{m} \sum_{l=-\infty m}^{\infty} x(l)\, z^{-l}$$

$$= z^{m}\left[\sum_{l=0}^{\infty} x(l)\, z^{-l} - \sum_{l=0}^{m-1} x(l)\, z^{-l}\right]$$

$$= z^{m}\left[X(z) - \sum_{l=0}^{m-1} x(l)\, z^{-l}\right]$$

$$= z^{m}\left[X(z) - \sum_{k=0}^{m-1} x(k)\, z^{-k}\right]$$

Qn. Find the z-transform of the following sequences and ROC using the properties of z-transform.

1) $x(n) = \delta(n-n_0)$.

$\delta(n) \xleftrightarrow{z} 1$

By appling time shifting property we get

$\delta(n-n_0) \xleftrightarrow{z} \underline{\underline{z^{-n_0}}}$ ROC: $|z|>0$

2) $x(n) = u(n-n_0)$.

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$

By applying time shifting property

$u(n-n_0) \xleftrightarrow{z} z^{-n_0}\frac{z}{z-1} = \frac{z^{-(n_0-1)}}{z-1}$ ROC: $1<|z|<\infty$

3) $x(n) = a^{n+1}u(n+1)$

$a^n u(n) \xleftrightarrow{z} \frac{z}{z-a}$.

$a^{n+1}u(n+1) \xleftrightarrow{z} \frac{z^1 z}{z-a} = \frac{z^2}{z-a}$ ROC: $|a|<|z|<\infty$

4) $x(n) = a^{n-1}u(n-1)$

$a^n u(n) \xleftrightarrow{z} \frac{z}{z-a}$.

$a^{n-1}u(n-1) \xleftrightarrow{z} \frac{z^{-1}z}{z-a} = \frac{1}{z-a}$.   ROC: $|a|<|z|<\infty$

5) $x(n) = (1/2)^n u(-n)$

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$

$u(-n) \xleftrightarrow{z} \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}$.

By using multiplication by exponential seq. property,

$x(n) = a^n x(n) \xleftrightarrow{z} X(z) = X(a^{-1}z)$

$(1/2)^n u(-n) \xleftrightarrow{z} \frac{1}{1-(\frac{1}{2})^{-1}z} = \frac{1}{1-2z}$ ROC: $|2||z|<1$

ROC: $|z| < 1/2$

6) $x(n) = n\,u(n)$.

$$u(n) \xleftrightarrow{Z} \frac{z}{z-1}$$

By using multiplication by 'n' property.

$$n\,u(n) \xleftrightarrow{Z} -z\frac{d}{dz}\frac{z}{z-1} = -z\frac{(z-1).1-z}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}.$$

$$\boxed{n\,u(n) \xleftrightarrow{Z} \frac{z}{(z-1)^2}}$$

7) Show that $u(n) * u(n-1) = n\,u(n)$

$$u(n) \xleftrightarrow{Z} \frac{z}{z-1} \qquad u(n-1) \xleftrightarrow{Z} \frac{1}{z-1}$$

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(z) \cdot X_2(z)$$

$$\therefore u(n) * u(n-1) \xleftrightarrow{Z} \frac{z}{z-1} \cdot \frac{1}{z-1} = \frac{z}{(z-1)^2}.$$

W.K.T $n\,u(n) \xleftrightarrow{Z} \frac{z}{(z-1)^2}$.

$\therefore u(n) * u(n-1) = n\,u(n)$.

8) $x(n) = \underbrace{n(-1/4)^n u(n)}_{x_1(n)} * \underbrace{(-1/6)^{-n} u(-n)}_{x_2(n)}$.

$$(-1/4)^n u(n) \xleftrightarrow{Z} \frac{z}{z+\frac{1}{4}}.$$

By using multiplication property

$$x_1(n) = n(-1/4)^n u(n) \xleftrightarrow{Z} X_1(z) = -z\frac{d}{dz}\frac{z}{z+\frac{1}{4}}$$

$$X_1(z) = -z\frac{(z+\frac{1}{4})-z}{(z+\frac{1}{4})^2} = \frac{-\frac{1}{4}z}{(z+\frac{1}{4})^2}. \qquad \text{ROC: } |z| > 1/4$$

$$(-1/6)^n u(n) \xleftrightarrow{Z} \frac{z}{z+\frac{1}{6}}$$

By using time reversal property

$$(-1/6)^{-n} u(-n) \xleftrightarrow{Z} \frac{z^{-1}}{z^{-1}+\frac{1}{6}} = \frac{1}{1+1/6\,z}$$

$$= \frac{+6}{z+6}. \qquad \text{ROC: } |z| < 6$$

$$X(z) = \frac{-\frac{1}{4}z}{(z+\frac{1}{4})^2} \cdot \frac{+6}{(z+6)}$$

$4(z^2 - \frac{1}{4}z - z + \frac{1}{4})$

$$= \frac{-1.5z}{(z+\frac{1}{4})^2 (z+6)} \quad //$$

ROC : $\frac{1}{4} < |z| < 6$

Qn. Find the initial and final values of the following function.

$$x(z) = \frac{z}{4z^2 - 5z + 1}$$

ROC : $|z| > 1$

initial value ; $x(0) = \lim_{z\to\infty} X(z)$

$$= \lim_{z\to\infty} \frac{z}{4z^2 - 5z + 1}$$

$$= \lim_{z\to\infty} \frac{\cancel{z}}{z^{\cancel{2}}(4 - \frac{5}{z} + \frac{1}{z^2})}$$

$1 - \frac{1}{4}$

$\frac{4-1}{4}$

$$= 0$$

Final value : $x(\infty) = \lim_{z\to 1} (z-1) X(z)$

$$= \lim_{z\to 1} z-1 \frac{z}{4z^2 - 5z - 1} = \lim_{z\to 1} \cancel{z-1} \frac{z}{4(\cancel{z-1})(z - \frac{1}{4})}$$

$$= \frac{1}{4(1 - \frac{1}{4})}$$

$$= \frac{1}{4 \times 3/4} = \frac{1}{3} //$$

Qn. Find $x(\infty)$, if $X(z) = \frac{z+1}{(z-0.6)^2}$

$$x(\infty) = \underset{z \to 1}{Lt} (z-1)\, X(z)$$

$$= \underset{z \to 1}{Lt} (z-1) \frac{(z+1)}{(z-0.6)^2} = 0$$

Qn. Find $x(\infty)$ if $X(z) = \frac{(z+2)}{4(z-1)(z+0.7)}$

$$x(\infty) = \underset{z \to 1}{Lt} \cancel{(z-1)} \frac{(z+2)}{4\cancel{(z-1)}(z+0.7)}$$

$$= \frac{3}{4 \times 1.7} = \frac{3}{6.8}$$

Inverse System

The inverse z-transform of the $X(z)$ is defined as

$$x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz.$$

Inverse z-transform:

There are four methods that are often used for the evaluation of inverse z-transform.

- → Long division method (power series method).
- → partial fraction method
- → Residue method
- → convolution method.

$$X(z) = \frac{N(z)}{D(z)}.$$

* For getting right sided seq., the N(z) and D(z) must be put in descending power of z before performing long division.

* For getting left sided seq., the N(z) and D(z) must be put in ascending power of z before performing long division.

1) long division method:

Qn. Using long division method, find the inverse z-transform of $X(z) = \frac{z}{2z^2-3z+1}$; ROC : $|z| > 1$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \rightarrow ①$$

Since ROC : $|z|>1$, $x(n)$ must be a right sided seq.

$$\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \\
2z^2-3z+1 & z \\
 & (-)z \overset{(+)}{-} \frac{3}{2} \overset{(-)}{+} \frac{1}{2}z^{-1} \\
\hline
 & \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & (-)\frac{3}{2} \overset{(+)}{-} \frac{9}{4}z^{-1} \overset{(-)}{+} \frac{3}{4}z^{-2} \\
\hline
 & \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & (-)\frac{7}{4}z^{-1} \overset{(+)}{-} \frac{21}{8}z^{-2} \overset{(-)}{+} \frac{7}{8}z^{-3} \\
\hline
 & \vdots
\end{array}$$

$X(z)$ = Quotient of long division

$$= \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \rightarrow ②$$

Comparing ① and ②.

$$x(n) = \left(\underset{\uparrow}{0}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots\right)$$

Qn. using long division method, find the inverse z-transform of $X(z) = \frac{z}{2z^2-3z+1}$, ROC: $|z| < \frac{1}{2}$.

Since ROC: $|z| < \frac{1}{2}$, $x(n)$ must be a left sided seq. For getting a left sided seq. the N(z) & D(z) must be put in ascending powers of z.

$$X(z) = \sum_{n=-\infty}^{0} x(n) z^{-n} = \cdots\cdots + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) \rightarrow ①$$

$$\begin{array}{r|l} & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\ \hline 1 - 3z + 2z^2 & z \\ & \underline{{}^{(-)}z \overset{(+)}{-} 3z^2 \overset{(-)}{+} 2z^3} \\ & \quad 3z^2 - 2z^3 \\ & \quad \underline{{}^{(-)}3z^2 \overset{(+)}{-} 9z^3 \overset{(-)}{+} 6z^4} \\ & \qquad 7z^3 - 6z^4 \\ & \qquad \underline{{}^{(-)}7z^3 \overset{(+)}{-} 21z^4 \overset{(-)}{+} 14z^5} \\ & \qquad\quad 15z^4 - 14z^5 \\ & \qquad\quad \vdots \end{array}$$

$X(z) =$ Quotient of long division.

$$= \cdots\cdots + 15z^4 + 7z^3 + 3z^2 + z \longrightarrow ②$$

Comparing ① and ②

$$x(n) = (\cdots\cdots, 15, 7, 3, 1, \underset{\uparrow}{0})$$

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Qn. By using long division method determine the inverse z-transform of $X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$ when a) x(n) is causal and b) x(n) is anticausal.

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}} = \frac{z^2+2z}{z^2-2z+1} = \frac{N(z)}{D(z)}$$

a) For getting ~~x(n)~~ causal causal seq, N(z) & D(z) must be put in descending power of z.

$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \rightarrow ①$

$$\begin{array}{r|l} & 1+4z^{-1}+7z^{-2}+10z^{-3}+\cdots \\ \hline z^2-2z+1 & z^2+2z \\ & \underset{(-)}{z^2} \overset{(+)}{-2z} \overset{(-)}{+1} \\ \hline & 4z-1 \\ & \underset{(-)}{4z} \overset{(+)}{-8} \overset{(-)}{+4z^{-1}} \\ \hline & 7-4z^{-1} \\ & \underset{(-)}{7} \overset{(+)}{-14z^{-1}} \overset{(-)}{+7z^{-2}} \\ \hline & 10z^{-1}-7z^{-2} \\ & \vdots \end{array}$$

X(z) = Quotient of long division.

$$= 1+4z^{-1}+7z^{-2}+10z^{-3}+\cdots \rightarrow ②$$

Comparing ① and ②

$$x(n) = (\underset{\uparrow}{1}, 4, 7, 10 \cdots)$$

For getting anti-causal seq., the N(z) and D(z) must be put in ascending powers of z.

$$X(z) = \sum_{n=-\infty}^{0} x(n) z^{-n} = \dots x(-4) z^{4} + x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) \rightarrow ①$$

$$
\begin{array}{r|l}
 & 2z + 5z^2 + 8z^3 + 11z^4 + \cdots \\
\hline
1 - 2z + z^2 & 2z + z^2 \\
 & \underline{\overset{(-)}{2z} \overset{(+)}{-} 4z^2 \overset{(-)}{+} 2z^3} \\
 & 5z^2 - 2z^3 \\
 & \underline{\overset{(-)}{5z^2} \overset{(+)}{-} 10z^3 \overset{(-)}{+} 5z^4} \\
 & 8z^3 - 5z^4 \\
 & \underline{\overset{(-)}{8z^3} \overset{(+)}{-} 16z^4 \overset{(-)}{+} 8z^5} \\
 & 11z^4 - 8z^5 \\
 & \vdots
\end{array}
$$

X(z) = Quotient of long division

$$= \dots + 11z^4 + 8z^3 + 5z^2 + 2z \rightarrow ①$$

Comparing ① and ②

$$x(n) = (\dots 11, 8, 5, 2, \underset{\uparrow}{0})$$

H.W

Using long division method find the inv. z-transform of $X(z) = \frac{z+1}{z^2 - 8z + 2}$ when a) x(n) is causal b) x(n) is anti-causal

Ans: a) $x(n) = (\underset{\uparrow}{0}, 1, 4, 10, 22)$

b) $x(n) = (\dots 1.81, 1.62, 1.25, \underset{\uparrow}{0.5})$

1) Partial fraction method:

	$X(z)$	$x(n)$		
1)	$\frac{z}{z-a}$	$a^n u(n)$	$\lvert z\rvert > \lvert a\rvert$	$\leftarrow$ causal (if $x(n)$ is causal)
2)	$\frac{z}{z-a}$	$-a^n u(-n-1)$	$\lvert z\rvert < \lvert a\rvert$	$\leftarrow$ anti causal (if $x(n)$ is anticausal)
3)	$\frac{z}{z-1}$	$u(n)$	$\lvert z\rvert > 1$	
4)	$\frac{z}{(z-a)^2}$	$n\,a^{n-1}u(n)$		
5)	$\frac{z}{(z-a)^3}$	$\frac{n(n-1)}{2!}a^{n-2}u(n)$.		

Qn. Find the inverse z-transform of $X(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})}$

ROC: $|z|>2$

1)

$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} = \frac{z(z-\frac{1}{3})}{(z-1)(z+2)}.$$

2) find $\frac{X(z)}{z}$.

$$\frac{X(z)}{z} = \frac{(z-\frac{1}{3})}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}.$$

$$z-\frac{1}{3} = A(z+2) + B(z-1)$$

Put $z=1 \Rightarrow 1-\frac{1}{3} = 3A \Rightarrow \frac{2}{3} = 3A \Rightarrow A = 2/9.$

Put $z=-2 \Rightarrow -2-\frac{1}{3} = -3B \Rightarrow -\frac{7}{3} = -3B \Rightarrow B = 7/9.$

$$\therefore \frac{X(z)}{z} = \frac{2}{9}\frac{1}{z-1} + \frac{7}{9}\frac{1}{z+2}.$$

$$X(z) = \frac{2}{9}\frac{z}{z-1} + \frac{7}{9}\frac{z}{z+2}$$

Taking inverse z-transform:

$$x(n) = \frac{2}{9}u(n) + \frac{7}{9}(-2)^n u(n)$$

Qn. Find the inverse z-transform of $X(z) = \frac{7z-23}{(z-3)(z-4)}$. ROC: $|z|>4$

$$X(z) = \frac{7z-23}{(z-3)(z-4)}$$

$$\frac{X(z)}{z} = \frac{7z-23}{z(z-3)(z-4)} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$7z-23 = A(z-3)(z-4) + Bz(z-4) + Cz(z-3).$$

Put $z=0 \Rightarrow A = -\frac{23}{12}$, Put $z=3 \Rightarrow B = \frac{2}{3}$ Put $z=4 \Rightarrow C = \frac{5}{4}$.

$$\frac{X(z)}{z} = -\frac{23}{12}\frac{1}{z} + \frac{2}{3}\frac{1}{z-3} + \frac{5}{4}\frac{1}{z-4}$$

$$X(z) = -\frac{23}{12} + \frac{2}{3}\frac{z}{z-3} + \frac{5}{4}\frac{z}{z-4}$$

Taking inverse z-transform.

$$x(n) = -\frac{23}{12}\delta(n) + \frac{2}{3}3^n u(n) + \frac{5}{4}4^n u(n).$$

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Qn. Determine the causal signal $x(n)$ having the z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}.$$

$1 - 2z^{-1} + z^{-2}$

$z^2 - 2z + 1$

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}.$$

$$\cancel{z^2 = A(z-1)(z-1)^2 + B(z+1)(z-1)^2 + C(z+1)(z-1)}$$

~~Put z=1 ⇒ 1=~~

$$\frac{z^2}{(z+1)\cancel{(z-1)^2}} = \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)\cancel{(z-1)^2}}$$

Put $z=1 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

Put $z=-1 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$

Equating coeff. of $z^2 \Rightarrow 1 = A+B \Rightarrow B = 1 - A = 1 - \frac{1}{4}$

$$= \frac{4-1}{4} = 3/4$$

$$\therefore \frac{X(z)}{z} = \frac{18}{4}\frac{1}{(z+1)} + \frac{3}{4}\frac{1}{(z-1)} + \frac{1}{2}\frac{1}{(z-1)^2}$$

$$X(z) = \frac{1}{4}\frac{z}{(z+1)} + \frac{3}{4}\frac{z}{(z-1)} + \frac{1}{2}\frac{z}{(z-1)^2}$$

Taking inverse z-transform.

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2} n (1)^{n-1} u(n)$$

$$= \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2} n u(n)$$

Qn. using partial fraction method find the inverse z-transform of $X(z) = \frac{z(z^2+z-30)}{(z-2)(z-4)^3}$, ROC $|z|>4$

$$X(z) = \frac{z(z-5)(z+6)}{(z-2)(z-4)^3}$$

$$\frac{X(z)}{z} = \frac{(z-5)(z+6)}{(z-2)(z-4)^3} = \frac{A}{z-2} + \frac{B}{z-4} + \frac{C}{(z-4)^2} + \frac{D}{(z-4)^3}$$

$A_1 = A$
$A_4 = B$
$A_3 - C$
$A_2 - D$

Put $z=2 \Rightarrow A = 3$, Put $z=4 \Rightarrow C = -5$,

Compare the coeff. of $z^2 \Rightarrow$ $C - 10B = 37$

compare the coeff. of $z \Rightarrow B$. $6C - 32B = 138$

$C = 7$ and $B = -3$

$$X(z) = 3\frac{z}{z-2} - 3\frac{z}{z-4} + 7\frac{z}{(z-4)^2} - 5\frac{z}{(z-4)^3}$$

Taking inverse

$$x(n) = 3\, 2^n u(n) - 3\, 4^n u(n) + 7 n\, 4^{n-1} u(n) - 5 \frac{n(n-1)}{2} 4^{n-2} u(n)$$

$$= 3\, 2^n u(n) - 3\, 4^n u(n) + \frac{7}{4} n\, 4^n u(n) - \frac{5}{32} n(n-1) 4^n u(n)$$

Determine the inverse Z-transform of $X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(1-3Z^{-1})}$ for all possible ROCs.

$$X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(1-3Z^{-1})} = \frac{5Z}{(Z-2)(Z-3)}$$

$$\frac{X(Z)}{Z} = \frac{5}{(Z-2)(Z-3)} = \frac{A}{Z-2} + \frac{B}{Z-3}$$

$5 = A(Z-3) + B(Z-2)$

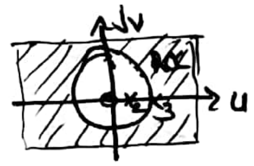
Put $Z=2 \Rightarrow 5 = -A \Rightarrow A = -5$, Put $Z=3 \Rightarrow 5 = B \Rightarrow B = 5$

$$\therefore \frac{X(Z)}{Z} = -5\frac{1}{Z-2} + 5\frac{1}{Z-3}.$$

$$X(Z) = -5\frac{Z}{Z-2} + 5\frac{Z}{Z-3}. \longrightarrow (1)$$

When the ROC is $|Z|>3$, then $x(n)$ is causal and all the two terms in eqn (1) are causal terms.

$\therefore x(n) = -5\,2^n u(n) + 5\,3^n u(n)$ //

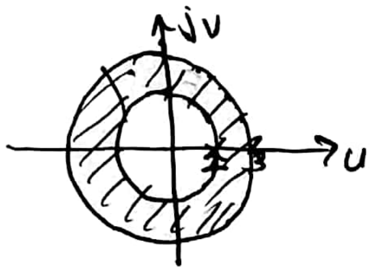


When the ROC is $|Z|<2$, then the signal $x(n)$ is anticausal and all the two terms in eqn (1) are anticausal terms. $\therefore x(n) = -5.(-)2^n u(-n-1) + 5.(-)3^n u(-n-1)$

$= 5\,2^n u(-n-1) - 5\,3^n u(-n-1).$

When the ROC is $2<|Z|<3$, then the signal is two sided. The pole $Z=2$ provides causal term and the pole $Z=3$ provides the anticausal term.

$\therefore x(n) = -5\,2^n u(n) - 5\,3^n u(-n-1)$



Contd.

Qn. Find the inverse z-transform of $x(z) = \frac{z}{3z^2-4z+1}$ for all possible ROCs.

$= \frac{z}{(z^2-\frac{4}{3}z+\frac{1}{3})}$

$$x(z) = \frac{z}{(3z^2-4z+1)} = \frac{z}{(z-1)(z-1/3)}$$

$1-\frac{1}{3} \quad \frac{3-1}{3} = 2/3$

$1/3 - 1 \quad \frac{-2}{3} = 3,$

$$\frac{x(z)}{z} = \frac{1}{(z-1)(z-\frac{1}{3})} = \frac{A}{(z-1)} + \frac{B}{z-1/3}$$

$1 = A(z-\frac{1}{3}) + B(z-1)$

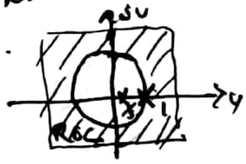
Put $z=1 \Rightarrow 1 = (1-\frac{1}{3})A \Rightarrow 1 = \frac{2}{3}A \Rightarrow A = 3/2$.

Put $z = 1/3 \Rightarrow 1 = B(1/3-1) \Rightarrow 1 = -\frac{2}{3}B \Rightarrow B = -3/2$

$\therefore \frac{x(z)}{z} = \frac{3}{2}\frac{1}{z-1} - \frac{3}{2}\frac{1}{z-1/3}$.

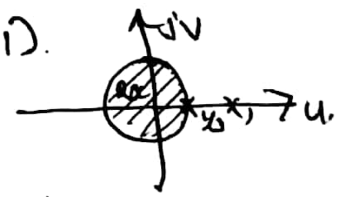
$x(z) = \frac{3}{2}\frac{z}{z-1} - \frac{3}{2}\frac{z}{z-1/3}$

When ROC is $|z|>1$, then the s/l $x(n)$ is causal and all the two terms are causal terms.



$\therefore x(n) = \frac{3}{2}u(n) - \frac{3}{2}(1/3)^n u(n)$.

When ROC is $|z| < \frac{1}{3}$, then the s/l $x(n)$ is anticausal and all the two terms are anticausal terms.



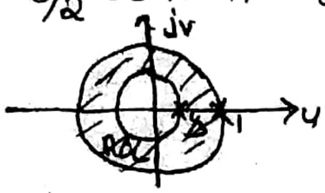
$\therefore x(n) = \frac{3}{2} -(1)^n u(-n-1) - \frac{3}{2}(-)(1/3)^n u(-n-1)$.

$= -3/2\, u(-n-1) + \frac{3}{2}(1/3)^n u(-n-1)$

When ROC is $1/3 < |z| < 1$, then the s/l $x(n)$ is two sided. The pole $z = 1/3$ provides causal term and the pole $z=1$ provides the anticausal term.

$\therefore x(n) = 3/2 -(1)^n u(-n-1) - \frac{3}{2}(1/3)^n u(n)$

$= -3/2\, u(-n-1) - 3/2\,(1/3)^n u(n)$



Residue method:

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz = \sum \text{residues of } X(z)\, z^{n-1} \text{ at poles within } c.$$

Cauchy residue theorem:

Let $f(z)$ be a function of the complex variable z and c be a closed path in the z-plane. If the derivative $\frac{d}{dz} f(z)$ exists on and inside the c and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{z - z_0}\, dz = f(z)\Big|_{z=z_0}$$
$$= f(z_0).$$

If the $(k+1)^{th}$ order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{(z-z_0)^k}\, dz = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}} f(z)\Big|_{z=z_0}$$

Qn. By using residue method, find the inverse z-transform of $X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$, ROC : $|z| > 2$

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{z(z+3)}{z^2+3z+2} = \frac{z(z+3)}{(z+1)(z+2)}$$

$x(n) = \sum$ residues of $X(z)\, z^{n-1}$ at poles within c.

$= \sum$ residues of $\frac{z(z+3)}{(z+1)(z+2)}\, z^{n-1}$ at poles within c.

$= \sum$ residues of $\frac{(z+3)z^n}{(z+1)(z+2)}$ at poles $z=-1$ and $z=-2$ within c.

$=$ residue of $\frac{(z+3)z^n}{(z+1)(z+2)}$ at pole $z=-1$ + residue of $\frac{(z+3)z^n}{(z+1)(z+2)}$ at pole $z=-2$

$$= \left.\frac{\cancel{(z+1)}(z+3)z^n}{\cancel{(z+1)}(z+2)}\right|_{z=-1} + \left.\frac{\cancel{(z+2)}(z+3)z^n}{(z+1)\cancel{(z+2)}}\right|_{z=-2}$$

$$= \left.\frac{(z+3)z^n}{(z+2)}\right|_{z=-1} + \left.\frac{(z+3)z^n}{(z+1)}\right|_{z=-2}$$

$$= \frac{2(-1)^n}{1} + \frac{1\cdot(-2)^n}{-1} = \left[2(-1)^n - (-2)^n\right]u(n).$$

Qn. Find the inverse z-transform of $X(z) = \frac{z^2+z}{(z-1)(z-3)}$, ROC: $|z|>3$.

$$X(z) = \frac{z^2+z}{(z-1)(z-3)} = \frac{z(z+1)}{(z-1)(z-3)}$$

$x(n) = \sum$ residues of $X(z)\,z^{n-1}$ at poles within C.

$= \sum$ residues of $\frac{z(z+1)}{(z-1)(z-3)} z^{n-1}$ at poles within C

$= \sum$ residues of $\frac{(z+1)z^n}{(z-1)(z-3)}$ at poles $z=1$ & $z=3$ within C

$=$ residue of $\frac{(z+1)z^n}{(z-1)(z-3)}$ at pole $z=1$ + residue of $\frac{(z+1)z^n}{(z-1)(z-3)}$ at pole $z=3$

$$= \left.\frac{\cancel{(z-1)}(z+1)z^n}{\cancel{(z-1)}(z-3)}\right|_{z=1} + \left.\frac{\cancel{(z-3)}(z+1)z^n}{(z-1)\cancel{(z-3)}}\right|_{z=3}$$

$$= \left.\frac{(z+1)z^n}{z-3}\right|_{z=1} + \left.\frac{(z+1)z^n}{z-1}\right|_{z=3}$$

$$= \frac{2\cdot 1^n}{-2} + \frac{4\cdot 3^n}{2}$$

$$= -1^n + 2\cdot 3^n$$

$$= \left[2\cdot 3^n - 1^n\right]u(n).$$

Qn. Find $x(n)$ if $X(z) = \frac{2z^{-1}}{(1-\frac{1}{4}z^{-1})^2}$; ROC $|z| > \frac{1}{4}$

$$X(z) = \frac{2z^{-1}}{(1-\frac{1}{4}z^{-1})^2} = \frac{2z}{(z-\frac{1}{4})^2}$$

$$X(z)\, z^{n-1} = \frac{2z}{(z-\frac{1}{4})^2} z^{n-1} = \frac{2z^n}{(z-\frac{1}{4})^2}$$

$2z^n \rightarrow f(z)$.

$$x(n) = \frac{1}{2\pi j} \oint_c \frac{f(z)}{(z-z_0)^k} dz = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} f(z) \Big|_{z=z_0}$$

$$\frac{1}{2\pi j} \oint_c \frac{2z^n}{(z-\frac{1}{4})^2} dz = \frac{d}{dz} f(z) \Big|_{z=1/4}$$

$$= \frac{d}{dz} 2z^n \Big|_{z=1/4} = 2n z^{n-1} \Big|_{z=1/4}$$

$$= 2n (1/4)^{n-1} u(n)$$

Qn. Find $x(n)$ if $X(z) = \frac{z^2}{(z-\frac{1}{2})^2}$ ROC : $|z| > \frac{1}{4}$

$$X(z)\, z^{n-1} = \frac{z^2 z^{n-1}}{(z-1/2)^2} = \frac{z^{n+1}}{(z-1/2)^2}$$

$z^{n+1} \rightarrow f(z)$.

$$x(n) = \frac{1}{2\pi j} \oint \frac{z^{n+1}}{(z-1/2)^2} dz = \frac{d}{dz} z^{n+1} \Big|_{z=1/2}$$

$$= (n+1) z^n \Big|_{z=1/2}$$

$$= (n+1)(1/2)^n u(n)$$

H.W 1) Find $x(n)$. if $X(z) = \frac{z^2+z}{(z-1)(z-3)}$ ROC : $|z| > 3$.

Ans : $2(3)^n u(n) - u(n)$

2) Find $x(n)$ if $X(z) = \frac{1-1/4\, z^{-1}}{1-1/9\, z^{-1}}$ ROC: $|z| > 1/3$.

Ans: $\frac{1}{8}(1/3)^n + \frac{7}{8}(-1/3)^n u(n)$.

Qn. Determine the inverse z-transform of $X(z) = \dfrac{z^{-1}}{(1-2z^{-1})(1-3z^{-1})}$

ROC: $2 < |z| < 3$

$$X(z) = \frac{z}{(z-2)(z-3)}$$

from the ROC, we can see that $x(n)$ is a two sided seq.

right sided $\rightarrow u(n)$

left sided $\rightarrow u(-n-1)$

$x(n) = -\sum$ residues of $X(z)\,z^{n-1}$ at pole $z=3$.

$+ \sum$ residue of $X(z)\,z^{n-1}$ at pole $z=2$.

$= -$ residue of $\dfrac{z^n}{(z-2)(z-3)}$ at pole $z=3$ + residue of $\dfrac{z^n}{(z-2)(z-3)}$ at pole $z=2$

$$= -\left.\frac{\cancel{(z-3)}\,z^n}{(z-2)\cancel{(z-3)}}\right|_{z=3} + \left.\frac{\cancel{(z-2)}\,z^n}{\cancel{(z-2)}(z-3)}\right|_{z=2}.$$

$$= \underbrace{-\frac{3^n}{1}}_{\text{left sided}} + \underbrace{\frac{2^n}{-1}}_{\text{right sided.}}$$

$$= -(3)^n u(-n-1) - (2)^n u(n)$$

Convolution method:

Convolution property: $x(n) = x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$

from the property, we know that the convolution of $x_1(n)$ and $x_2(n)$ is the inverse z-transform of $X(z)$. Thus $x(n)$ can be obtained by convolving $x_1(n)$ & $x_2(n)$.

Qn. Find the inverse z-transform of $X(z) = \dfrac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$

ROC: $|z| > 2$ using convolution method.

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{z(z+3)}{z^2+3z+2}.$$

$$X(z) = \frac{z}{(z+1)}\cdot\frac{z+3}{(z+2)}$$

Let $X(z) = X_1(z)\cdot X_2(z)$.

when $X_1(z) = \frac{z}{z+1} \Longrightarrow x(n) = (-1)^n u(n)$.

and $X_2(z) = \frac{z+3}{z+2} = \frac{z}{z+2} + z^{-1}\frac{3z}{z+2}$.

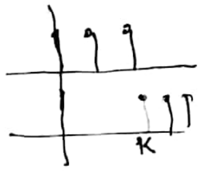
$$x_2(n) = (-2)^n u(n) + 3\cdot(-2)^{n-1}u(n-1).$$

$x(n) = x_1(n) * x_2(n)$

$= (-1)^n u(n) * \left[(-2)^n u(n) + 3(-2)^{n-1}u(n-1)\right]$

$= \underbrace{(-1)^n u(n) * (-2)^n u(n)}_{''} + (-1)^n u(n) * 3(-2)^{n-1}u(n-1).$

$= \sum_{k=-\infty}^{+\infty} (-1)^k u(k)\,(-2)^{n-k}\, u(n-k).$



$= \sum_{k=0}^{n} (-1)^k (-2)^n \cdot (-2)^{-k}$

$= \sum_{k=0}^{n} (0.5)^k (-2)^n = (-2)^n \sum_{k=0}^{n} (0.5)^k = (-2)^n \frac{1-(0.5)^{n+1}}{1-0.5}.$

$= (-2)^n\left(\frac{1-0.5^n\cdot 0.5}{0.5}\right) = (-2)^n(2-0.5^n)$

$= 2(-2)^n - (-1)^n.$

$(-1)^n u(n) * 3(-2)^{n-1}u(n-1) = 3\sum_{k=-\infty}^{\infty}(-1)^k u(k)(-2)^{n-1-k}\, u(n-1-k).$

$= 3\sum_{k=0}^{n-1}(-1)^k(-2)^{n-1}(-2)^{-k} = 3(-2)^{n-1}\sum_{k=0}^{n-1}(0.5)^k.$

$= 3(-2)^{n-1}\left[\frac{1-0.5^n}{1-0.5}\right] = 3(-2)^{n-1}\left[\frac{1-0.5^n}{0.5}\right]$

$= 6(-2)^{n-1} - 6(-2)^{n-1}(0.5)^n.$

$= 6(-2)^n\times\frac{1}{-2} - 6(-2)^n\frac{1}{-2}(0.5)^n.$

$= -3(-2)^n + 3(-1)^n.$

$x(n) = 2(-2)^n - (-1)^n + -3(-2)^n + 3(-1)^n.$

$= -(-2)^n + 2(-1)^n. \quad = \underbrace{\left[2(-1)^n - (-2)^n\right] u(n)}.$

Qn. Find the inverse z-transform of $X(z) = \dfrac{z^2}{(z-2)(z-4)}$ using convolution method.

$X(z) = \dfrac{z}{(z-2)} \cdot \dfrac{z}{(z-4)} = X_1(z) \cdot X_2(z).$

$X_1(z) = \dfrac{z}{z-2} \Rightarrow x_1(n) = 2^n u(n)$

$X_2(z) = \dfrac{z}{z-4} \Rightarrow x_2(n) = 4^n u(n)$

$n-k \geq 0.$

$n \geq k.$

$k \leq n.$

$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k).$

$= \sum_{k=-\infty}^{\infty} 2^k u(k)\, 4^{n-k} u(n-k)$

$= \sum_{k=0}^{n} 2^k 4^{n-k} = \sum_{k=0}^{n} 2^k 4^n . 4^{-k} = 4^n \sum_{k=0}^{n} 2^k . \left(\frac{1}{4}\right)^k$

$= 4^n \sum_{k=0}^{n} \left(2/4\right)^k = 4^n \sum_{k=0}^{n} (0.5)^k = 4^n \left[\dfrac{1-0.5^{n+1}}{1-0.5}\right]$

$= 4^n \left(\dfrac{1-0.5^n . 0.5}{0.5}\right) = 4^n \left[2 - 0.5^n\right]$

$= \left[2(4)^n - (2)^n\right] u(n).$

Qn - Find the inverse z-transform of $X(z) = \frac{z}{(z-1)(z-\frac{1}{2})}$ using convolution property. Also verify the answer using partial fraction and residue method.

By convolution

$$X(z) = \frac{z}{z-1} \cdot \frac{1}{z-\frac{1}{2}} = X_1(z) \cdot X_2(z).$$

$$X_1(z) = \frac{z}{z-1} \Rightarrow x_1(n) = u(n)$$

$$X_2(z) = \frac{1}{z-\frac{1}{2}} = z^{-1}\frac{z}{z-\frac{1}{2}} \Rightarrow x_2(n) = (\tfrac{1}{2})^{n-1}u(n-1).$$

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} u(k)\,(\tfrac{1}{2})^{n-1-k}\,u(n-1-k).$$

$$= \sum_{k=0}^{n-1} (\tfrac{1}{2})^{n-1-k} = \sum_{k=0}^{n-1} (\tfrac{1}{2})^{n-1} (\tfrac{1}{2})^{-k}$$

$$= (\tfrac{1}{2})^{n-1} \sum_{k=0}^{n-1} 2^k = (\tfrac{1}{2})^{n-1}\left[\frac{1-2^n}{1-2}\right]$$

$$= (\tfrac{1}{2})^{n-1}\left[2^n - 1\right] = 2^n(\tfrac{1}{2})^{n-1} - (\tfrac{1}{2})^{n-1}$$

$$= \left[2 - 2(\tfrac{1}{2})^n\right]u(n)$$

By partial fraction method: $\frac{X(z)}{z} = \frac{1}{(z-1)(z-\frac{1}{2})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$

$1 = A(z-\frac{1}{2}) + B(z-1)$, Put $z=1 \Rightarrow \frac{1}{2}A = 1 \Rightarrow A = 2$.

Put $z = \frac{1}{2} \Rightarrow -\frac{1}{2}B = 1 \Rightarrow B = -2$,

$$X(z) = 2\frac{z}{z-1} - 2\frac{z}{z-\frac{1}{2}}$$

Taking inv, $x(n) = 2u(n) - 2(\tfrac{1}{2})^n u(n) = \left[2 - 2(\tfrac{1}{2})^n\right]u(n)$

By residue method:

$x(n) = \sum$ residues of $\frac{z}{(z-1)(z-\frac{1}{2})} z^{n-1}$ at poles within C.

$= \sum$ residues of $\frac{z^n}{(z-1)(z-\frac{1}{2})}$ at poles $z=1$ & $z=\frac{1}{2}$ within C.

$=$ residue of $\frac{z^n}{(z-1)(z-\frac{1}{2})}$ at pole $z=1$ + residue of $\frac{z^n}{(z-1)(z-\frac{1}{2})}$ at pole $z=\frac{1}{2}$

$$= \frac{\cancel{(z-1)}\, z^n}{\cancel{(z-1)}(z-\frac{1}{2})}\Bigg|_{z=1} + \frac{\cancel{(z-\frac{1}{2})}\, z^n}{(z-1)\cancel{(z-\frac{1}{2})}}\Bigg|_{z=\frac{1}{2}} = \frac{1}{\frac{1}{2}} + \frac{(\frac{1}{2})^n}{-\frac{1}{2}} = \left[2 - 2(\tfrac{1}{2})^n\right]u(n)$$

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Analysis of LTI Systems:

The z-transform plays an important role in the analysis and design of discrete time LTI systems.

System function (transfer function) and impulse response:

Consider a discrete time LTI system having an impulse response h(n) as shown in fig. given below.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n).$$

Let us say it gives an output y(n) for an input x(n).

Then we have $y(n) = x(n) * h(n)$.

Taking z-transform on both sides.

$$Y(Z) = X(Z) \cdot H(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

H(Z) is called the system function or the transfer function of the LTI discrete time s/m and is defined as the ratio of the z-transform of the o/p sequence y(n) to the z-transform of the i/p sequence x(n), when the initial conditions are neglected.

The poles of the system are defined as the values of z for which the system function $H(Z) = \infty$ and the zeros of the system are the values of z for which the system function $H(Z) = 0$.

Relationship between transfer function and difference equation

consider an n^{th} order LTI DT s/m described by the difference eqn. $\sum_{K=0}^{N} a_K\, y(n-K) = \sum_{K=0}^{M} b_K\, x(n-K)$.

Taking z-transform on both sides.

$$\sum_{K=0}^{N} a_K z^{-K} Y(z) = \sum_{K=0}^{M} b_K z^{-K} X(z)$$

$$Y(z) \sum_{K=0}^{N} a_K z^{-K} = X(z) \sum_{K=0}^{M} b_K z^{-K}$$

$$\frac{Y(z)}{H(z)} = \frac{\sum_{K=0}^{M} b_K z^{-K}}{\sum_{K=0}^{N} a_K z^{-K}} = \frac{b_0 + b_1 z^{-1} + \cdots b_M z^{-m}}{a_0 + a_1 z^{-1} + \cdots a_N z^{-N}}$$

where $\frac{Y(z)}{X(z)} = H(z)$ is called the system function or transfer function of the system. The frequency response of a system is obtained by substituting $z = e^{j\omega}$ in $H(z)$.

Stability and causality:

The necessary and sufficient condition for a ~~causal~~ linear time invariant system to be BIBO stable is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

If the system is causal, $h(n) = 0$ for $n < 0$.

For a causal LTI s/m the condition for stability is

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

System function of a causal LTI s/m, $H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$.

magnitude

$$|H(z)| = \left| \sum_{n=0}^{\infty} h(n) z^{-n} \right| = \sum_{n=0}^{\infty} |h(n)| \, |z^{-n}|$$

The evaluation of $|H(z)|$ on unit circle yields.

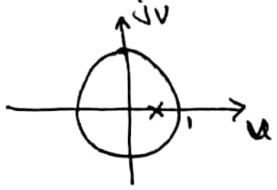
$$|H(z)| = \sum_{n=0}^{\infty} |h(n)| < \infty \quad (\because |z| = 1 \text{ for unit circle})$$

Therefore for a stable system, the ROC of a system function includes unit circle.

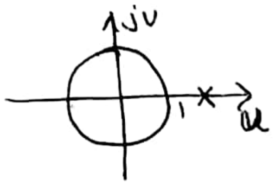
For a causal system, the ROC is exterior of the circle of radius 'r'. Since ROC cannot contain any pole of H(z), a causal LTI system is BIBO stable.

Show that for a stable s/m, the ROC of a s/m function includes the unit circle.

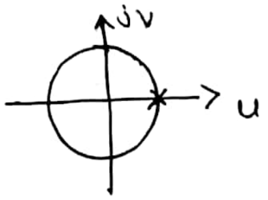
if and only if all the poles of $H(Z)$ are inside the unit circle.



$\rightarrow$ stable because a pole is inside the unit circle.



$\rightarrow$ unstable because pole is outside the unit circle.



$\rightarrow$ unstable because pole is in the unit circle

Problems on System function

1) Find the s/m function of the 1^{st} order difference equation

$$y(n) - 2y(n-1) = x(n) + x(n-1)$$

Taking z-transform on both sides.

$$Y(z) - 2z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$Y(z)[1-2z^{-1}] = X(z)[1+z^{-1}].$$

System function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1+z^{-1}}{1-2z^{-1}}$

2) Given $x(n) = u(n)$ and $y(n) = 2^n u(n)$. Find the system function and impulse response.

$$X(z) = \frac{z}{z-1} \quad \text{and} \quad Y(z) = \frac{z}{z-2}.$$

System function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\cancel{z}}{z-2} \times \dfrac{z-1}{\cancel{z}}$

$$H(z) = \frac{z-1}{z-2}.$$

Impulse response, $h(n)$:

$$\frac{H(z)}{z} = \frac{z-1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2} \Rightarrow z-1 = A(z-2) + Bz.$$

Put $z=0 \Rightarrow -1 = -2A \Rightarrow A = \frac{1}{2}$, Put $z=2 \Rightarrow 1 = 2B$, $B = \frac{1}{2}$

$\therefore \frac{H(z)}{z} = \frac{1}{2}\frac{1}{z} + \frac{1}{2}\frac{1}{z-2}$

$\therefore H(z) = \frac{1}{2} + \frac{1}{2}\frac{z}{z-2}$

Taking inverse, $h(n) = \frac{1}{2}\delta(n) + \frac{1}{2} 2^n u(n)$

3) An LTI s/m is described by the following s/m function

$H(z) = \frac{z+\frac{1}{2}}{(z-1)(z-\frac{1}{2})}$. Find the s/m response $y(n)$ to the i/p

$x(n) = 4^{-(n+2)} u(n)$.

$x(n) = 4^{-(n+2)} u(n) = 4^{-n} \cdot 4^{-2} u(n) = \frac{(\frac{1}{4})^n}{16} u(n)$

$X(z) = \frac{1}{16}\frac{z}{z-\frac{1}{4}}$

W.K.T $H(z) = \frac{Y(z)}{X(z)} \Rightarrow Y(z) = H(z) \cdot X(z)$

$Y(z) = \frac{z+\frac{1}{2}}{(z-1)(z-\frac{1}{2})} \times \frac{1}{16}\frac{z}{z-\frac{1}{4}}$

$\frac{Y(z)}{z} = \frac{1}{16}\frac{z+\frac{1}{2}}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{(z-1)} + \frac{B}{(z-\frac{1}{2})} + \frac{C}{z-\frac{1}{4}}$

$\frac{1}{16}(z+\frac{1}{2}) = A(z-\frac{1}{2})(z-\frac{1}{4}) + B(z-1)(z-\frac{1}{4}) + C(z-1)(z-\frac{1}{2})$

Put $z = \frac{1}{2} \Rightarrow \frac{1}{16} = B(\frac{-1}{2} \times \frac{1}{4}) \Rightarrow \frac{1}{16} = -\frac{1}{8}B \Rightarrow B = -\frac{1}{2}$

Put $z = \frac{1}{4} \Rightarrow \frac{1}{16} \times \frac{3}{4} = C(-3/4)(-\frac{1}{4}) \Rightarrow \frac{3}{16\times4} = \frac{3}{16}C \Rightarrow C = \frac{3}{16\times4} \times \frac{16}{3}$

$C = \frac{1}{4}$

Put $z = 1 \Rightarrow \frac{1}{16} \times \frac{3}{2} = A(\frac{1}{2})(\frac{3}{4}) \Rightarrow \frac{3}{16\times2} = \frac{3}{8}A \Rightarrow A = \frac{3}{16\times2} \times \frac{8}{3} = \frac{1}{4}$

$\therefore \frac{Y(z)}{z} = \frac{1}{4}\frac{1}{(z-1)} - \frac{1}{2}\frac{1}{(z-\frac{1}{2})} + \frac{1}{4}\frac{1}{(z-\frac{1}{4})}$

$Y(z) = \frac{1}{4}\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-\frac{1}{2}} + \frac{1}{4}\frac{z}{z-\frac{1}{4}}$

Taking inverse z-transform

$y(n) = \frac{1}{4}u(n) - \frac{1}{2}(\frac{1}{2})^n u(n) + \frac{1}{4}(\frac{1}{4})^n u(n)$.

system response

H.W

4) The i/p to the system $x(n) = u(-n-1) + \left(\frac{1}{2}\right)^n u(n)$. The z-transform of the s/m o/p is $Y(z) = \dfrac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}$. Determine the impulse response and o/p of the system.

$$X(z) = \frac{\frac{1}{2}z}{(1-z)(z-\frac{1}{2})}. \qquad Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} = \frac{-\frac{1}{2}z}{(z-\frac{1}{2})(z+1)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z}{(z-\frac{1}{2})(z+1)} \times \frac{(1-z)(z-\frac{1}{2})}{\frac{1}{2}z} = \frac{-(1-z)}{z+1} = \frac{z-1}{z+1}$$

impulse response $h(n)$:

$$\frac{H(z)}{z} = \frac{z-1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$$

$$\Rightarrow z-1 = A(z+1) + Bz$$

Put $z=0 \Rightarrow -1 = A$; put $z=-1 \Rightarrow -2 = -B \Rightarrow B = 2$.

$$\therefore \frac{H(z)}{z} = -\frac{1}{z} + 2\frac{1}{z+1}$$

$$H(z) = -1 + 2\frac{z}{z+1}$$

Taking inv. z-transform, impulse response, $h(n) = -\delta(n) + 2(-1)^n u(n)$

o/p of the system $y(n)$:

$$\frac{Y(z)}{z} = \frac{-\frac{1}{2}}{(z-\frac{1}{2})(z+1)} = \frac{A}{(z-\frac{1}{2})} + \frac{B}{z+1}$$

$$A = -\frac{1}{3} \text{ and } B = \frac{1}{3}.$$

$$\frac{Y(z)}{z} = -\frac{1}{3}\frac{1}{z-\frac{1}{2}} + \frac{1}{3}\frac{1}{z+1}$$

$$Y(z) = -\frac{1}{3}\frac{z}{z-\frac{1}{2}} + \frac{1}{3}\frac{z}{z+1}$$

Taking inverse z-transform,

o/p of the system, $y(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}(-1)^n u(n)$

Qn. A causal system is represented by $H(z) = \frac{z+2}{2z^2-3z+4}$

find the difference eqn. and the frequency response of the system.

Given $H(z) = \frac{z+2}{2z^2-3z+4}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z+2}{2z^2-3z+4} = \frac{\cancel{z^2}(z^{-1}+2z^{-2})}{\cancel{z^2}(2-3z^{-1}+4z^{-2})}$$

$$Y(z)(2-3z^{-1}+4z^{-2}) = X(z)(z^{-1}+2z^{-2}).$$

$$2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

Taking inverse z-transform.

$$2y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2x(n-2).$$

which is the required difference eqn.

Putting $z = e^{j\Omega}$ in $H(z)$, we get the frequency response $H(e^{j\Omega})$ of the system.

$$H(z) = \frac{z+2}{2z^2-3z+4}$$

freq. resp. $H(e^{j\Omega}) = \frac{e^{j\Omega}+2}{2e^{j2\Omega}-3e^{j\Omega}+4}$

$$= \frac{2+\cos\Omega + j\sin\Omega}{4+(2\cos 2\Omega + j2\sin 2\Omega - 3\cos\Omega - 3j\sin\Omega)}$$

$$= \frac{2+\cos\Omega + j\sin\Omega}{4+(2\cos 2\Omega - 3\cos\Omega) + j(2\sin 2\Omega - 3\sin\Omega)}$$

3a. plot the pole-zero pattern and determine which of the following systems are stable:

a) $y(n) = y(n-1) - 0.8\,y(n-2) + x(n) + x(n-2)$

b) $y(n) = 2y(n-1) - 0.8\,y(n-2) + x(n) + 0.8\,x(n-1)$

a) Given $y(n) = y(n-1) - 0.8\,y(n-2) + x(n) + x(n-2)$

Taking z-transform.

$$Y(z) = z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + z^{-2}X(z)$$

$$Y(z) - z^{-1}Y(z) + 0.8z^{-2}Y(z) = X(z) + z^{-2}X(z)$$

$$Y(z)\left[1 - z^{-1} + 0.8z^{-2}\right] = X(z)\left[1 + z^{-2}\right]$$

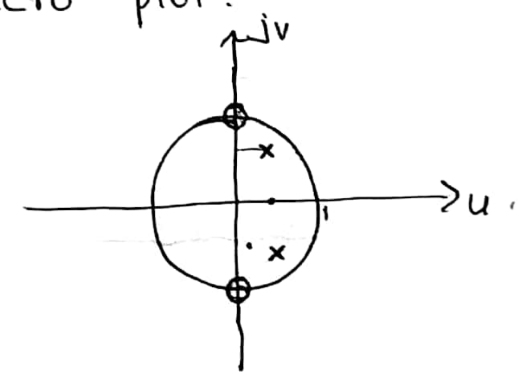
System function, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1+z^{-2}}{1-z^{-1}+0.8z^{-2}}$

$$= \frac{z^2+1}{z^2-z+0.8} = \frac{(z+j)(z-j)}{(z-0.5-j0.74)(z-0.5+j0.74)}$$

zeros of $H(z)$: $z = 1j$ and $z = -1j$

poles of $H(z)$: $z = 0.5 + j0.74$ and $z = 0.5 - j0.74$

pole-zero plot:



All the poles are inside the unit circle. Hence the system is stable.

b) Given $y(n) = 2y(n-1) - 0.8\,y(n-2) + x(n) + 0.8\,x(n-1)$.

$$Y(z) = 2z^{-1}Y(z) - 0.8\,z^{-2}Y(z) + X(z) + 0.8\,z^{-1}X(z)$$

$$Y(z) - 2z^{-1}Y(z) + 0.8\,z^{-2}Y(z) = X(z) + 0.8\,z^{-1}X(z)$$

$$Y(z)\left[1 - 2z^{-1} + 0.8\,z^{-2}\right] = X(z)\left[1 + 0.8\,z^{-1}\right]$$

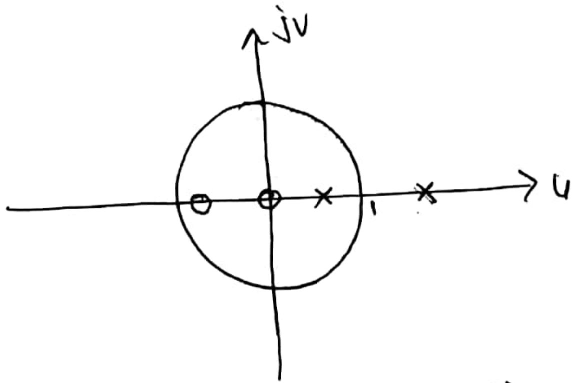
System function, $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.8\,z^{-1}}{1 - 2z^{-1} + 0.8\,z^{-2}}$

$$H(z) = \frac{z^2 + 0.8z}{z^2 + 2z + 0.8} = \frac{z(z+0.8)}{(z-1.44)(z-0.55)}$$

zeros of $H(z)$: $z = 0$ and $z = -0.8$

poles of $H(z)$: $z = 1.44$ and $z = 0.55$

pole - zero plot:



One pole is outside the unit circle

$\therefore$ the s/m is unstable.

Qn. Consider a causal discrete time LTI s/m with input $x(n) = (1/3)^n u(n) - 2(\frac{1}{3})^{n-1} u(n-1)$. and output $y(n) = (1/2)^n u(n)$. Determine the transfer function $(H(z))$, impulse response $h(n)$ and frequency response $H(e^{j\Omega})$ of the system.

Given
$$x(n) = \underbrace{(1/3)^n u(n)}_{\frac{z}{z-1/3}} - \underbrace{2(1/3)^{n-1} u(n-1)}_{2z^{-1}\frac{z}{z-1/3}.}$$

$$X(Z) = \frac{Z}{Z-\frac{1}{3}} - \frac{2}{Z-1/3} = \frac{Z-2}{Z-1/3}.$$

Given $y(n) = (1/2)^n u(n)$ $\therefore Y(Z) = \frac{Z}{Z-1/2}$

System function, $H(Z) = \frac{Y(Z)}{X(Z)} = \frac{Z}{Z-1/2} \times \frac{Z-2}{Z-1/3}$ $\frac{Z(Z-\frac{1}{3})}{(Z-2)(Z-1/2)}.$

$$H(Z) = \frac{Z(Z-2)}{(Z-1/2)(Z-1/3)}.$$

$1/2 - 2 = \frac{1-4}{2} = -3/2$

$1/2 - 1/3 \quad \frac{3-2}{6} = 1/6$

$1/3 - 2 = \frac{1-6}{3} = -5/3$

$1/3 - 1/2 = \frac{2-3}{6} = -1/6$

Impulse response, $h(n)$:

$$\frac{H(Z)}{Z} = \frac{Z-2}{(Z-1/2)(Z-1/3)} = \frac{A}{(Z-1/2)} + \frac{B}{(Z-1/3)}$$

$$Z-2 = A(Z-1/3) + B(Z-1/2)$$

Put $Z = 1/2 \Rightarrow -3/2 = 1/6 A \Rightarrow A = -3/2 \times 6 = -9.$

Put $Z = 1/3 \Rightarrow -5/3 = -1/6 B \Rightarrow B = -5/3 \times -6^2 = 10$

$$\therefore \frac{H(Z)}{Z} = -9\frac{1}{Z-1/2} + 10\frac{1}{Z-1/3}.$$

$$H(Z) = -9\frac{Z}{Z-1/2} + 10\frac{Z}{Z-1/3}$$

Taking inverse Z-transform

impulse response, $h(n) = -9(1/2)^n u(n) + 10(1/3)^n u(n)$

Frequency response: $H(e^{j\Omega})$.

$$H(Z) = \frac{Z(Z-2)}{(Z-1/2)(Z-1/3)}$$

$$H(e^{j\Omega}) = \frac{e^{j\Omega}(e^{j\Omega}-2)}{(e^{j\Omega}-1/2)(e^{j\Omega}-1/3)}$$

Qn. Determine the impulse response and step response of the causal system given below and discuss on stability. $y(n) - y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$

Given $y(n) - y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$

$$Y(z) - z^{-1}Y(z) - 2z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

$$Y(z)[1 - z^{-1} - 2z^{-2}] = X(z)[z^{-1} + 2z^{-2}].$$

System fun., $H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + 2z^{-2}}{1 - z^{-1} - 2z^{-2}}$

$$H(z) = \frac{z+2}{z^2 - z - 2} = \frac{z+2}{(z-2)(z+1)}$$

Impulse response, $h(n)$: $\frac{H(z)}{z} = \frac{z+2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{(z-2)} + \frac{C}{z+1}$

$$z+2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2).$$

Put $z=0 \Rightarrow 2 = A(-2)(1) \Rightarrow 2 = -2A \Rightarrow A = -1$

Put $z = 2 \Rightarrow 4 = 6B \Rightarrow B = 4/6 = 2/3.$

Put $z = -1 \Rightarrow 1 = 3C \Rightarrow C = 1/3$

\# unstable.

$$\therefore \frac{H(z)}{z} = -\frac{1}{z} + \frac{2}{3}\frac{1}{z-2} + \frac{1}{3}\frac{1}{z+1}$$

$$H(z) = -1 + \frac{2}{3}\frac{z}{z-2} + \frac{1}{3}\frac{z}{z+1}$$

Taking inverse

$$h(n) = -\delta(n) + \frac{2}{3}2^n u(n) + \frac{1}{3}(-1)^n u(n).$$

For step response, $x(n) = u(n). \Rightarrow X(z) = \frac{z}{z-1}$

output, $Y(z) = H(z).X(z) = \frac{z+2}{(z-2)(z+1)} \cdot \frac{z}{(z-1)}$

$\frac{Y(z)}{z} = \frac{z+2}{(z-2)(z+1)(z-1)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z-1}$ | $A = 4/3$, $B = 1/6$, $C = -3/2$

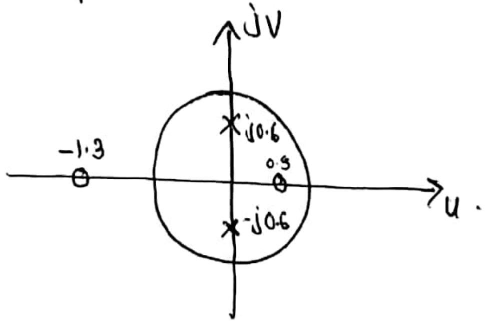
$\therefore y(n) = 4/3\, 2^n u(n) + \frac{1}{6}(-1)^n u(n) - 3/2\, u(n)$ //

Determining the frequency response from poles & zeros:

Step (1) from the pole-zeros, write the system function $H(z)$

Step (2) Find $H(z)\Big|_{z=e^{j\Omega}}$; we get $H(e^{j\Omega})$, is the frequency response.

Qn. Determine the frequency response from the following pole-zero plot.



Zeros : $z=-1.3$ and $z=0.5$

poles : $z=j0.6$ and $z=-j0.6$.

$$H(z)=\frac{(z+1.3)(z-0.5)}{(z-j0.6)(z+j0.6)}$$

frequency response : $$H(e^{j\Omega})=\frac{(e^{j\Omega}+1.3)(e^{j\Omega}-0.5)}{(e^{j\Omega}-j0.6)(e^{j\Omega}+j0.6)}$$

Qn. The zeros of $H(z)$ are $z=0$ and $z=-0.8$ and the poles of $H(z)$ are $z=1.4$ and $z=0.5$. Determine the frequency response of the s/m.

$$H(z)=\frac{z(z+0.8)}{(z-1.4)(z-0.5)}$$

freq. resp, $$H(e^{j\Omega})=\frac{e^{j\Omega}(e^{j\Omega}+0.8)}{(e^{j\Omega}-1.4)(e^{j\Omega}-0.5)}$$ //

Solution of LTI Systems described by the difference eqn:

$$\left.\begin{array}{l} y(n-1) \overset{z}{\longleftrightarrow} z^{-1}Y(z) + y(-1) \\ y(n-2) \overset{z}{\longleftrightarrow} z^{-2}Y(z) + z^{-1}y(-1) + y(-2) \end{array}\right\} \text{shifting property.}$$

$$\left.\begin{array}{l} y(-1) \\ y(-2) \end{array}\right\} \rightarrow \text{initial conditions.}$$

Steps: 1) For a given set of initial conditions, take the z-transform of both sides of the diff. eqn. to obtain the algebraic eqn. in $Y(z)$.

2) Solve the algebraic eqn for $Y(z)$

3) Take the inverse z-transform.

Qn. Determine the step response of the system $y(n) - \frac{1}{2}y(n-1) = x(n) - \frac{1}{2}x(n-1)$. Assume the initial conditions as $y(-1) = 1$

For step response
$x(n) = u(n)$
$X(z) = \frac{z}{z-1}$

$$y(n) - \tfrac{1}{2}y(n-1) = x(n) - \tfrac{1}{2}x(n-1)$$

Taking z-transform.

$$Y(z) - \frac{1}{2}z^{-1}Y(z) + y(-1) = X(z) - \frac{1}{2}z^{-1}X(z) + x(-0)$$

$$Y(z) - \frac{1}{2}\left[z^{-1}Y(z) + 1\right] = X(z) - \frac{1}{2}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] - \frac{1}{2} = \left[1 - \frac{1}{2}z^{-1}\right]X(z).$$

$$Y(z) = \frac{\frac{1}{2} + \left[1 - \frac{1}{2}z^{-1}\right]\frac{z}{z-1}}{\left[1 - \frac{1}{2}z^{-1}\right]}$$

$$= \frac{1}{2}\,\frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{z}{z-1}$$

$$Y(z) = \frac{1}{2}\,\frac{z}{z - \frac{1}{2}} + \frac{z}{z-1}$$

Taking inverse z-transform

$$y(n) = \frac{1}{2}\left(\tfrac{1}{2}\right)^n u(n) + u(n)$$

Qn. Determine the response of the s/m given by the difference eqn. $y(n) - 0.5\,y(n-1) = x(n)$, when the input is $x(n) = 5^n u(n)$ and the initial condition is $y(-1) = 2$.

Given $x(n) = 5^n u(n) \Rightarrow X(z) = \frac{z}{z-5}$.

$y(n) - 0.5\,y(n-1) = x(n)$.

Taking z- transform.

$Y(z) - 0.5\left[z^{-1}Y(z) + y(0)\right] = X(z)$.

$Y(z) - 0.5\left[z^{-1}Y(z) + 2\right] = X(z)$

$Y(z) - 0.5z^{-1}Y(z) - 1 = X(z)$.

$Y(z)\left[1 - 0.5z^{-1}\right] + 1 = X(z)$

$Y(z)\left[1 - 0.5z^{-1}\right] = X(z) + 1$

$Y(z)\left[1 - 0.5z^{-1}\right] = \frac{z}{z-5} + 1$

$$Y(z) = \frac{\frac{z}{z-5}}{1-0.5z^{-1}} + \frac{1}{1-0.5z^{-1}}$$

$$= \frac{z}{(z-5)(1-0.5z^{-1})} + \frac{1}{1-0.5z^{-1}}$$

$$= \underbrace{\frac{z^2}{(z-5)(z-0.5)}}_{Y_1(z)} + \underbrace{\frac{z}{z-0.5}}_{Y_2(z)}$$

$Y_2(z) \Rightarrow y_2(n) = 0.5^n u(n)$.

~~To find~~

$\cancel{Y_1}$ $y(n) = y_1(n) + y_2(n)$.

To find $y_1(n)$:

$$\frac{Y_1(z)}{z} = \frac{z}{(z-5)(z-0.5)} = \frac{A}{(z-5)} + \frac{B}{(z-0.5)}.$$

$A = 10/9$

$B = -1/9$

$1 - 1/9$

$\frac{9-1}{9}$

$$\frac{Y_1(z)}{z} = \frac{10}{9}\frac{1}{(z-5)} - \frac{1}{9}\frac{1}{(z-0.5)}$$

$$Y_1(z) = \frac{10}{9}\frac{z}{z-5} - \frac{1}{9}\frac{z}{z-0.5}$$

Taking inv. z- transform

$y_1(n) = 10/9\; 5^n u(n) - 1/9\,(0.5)^n u(n)$

$y(n) = y_1(n) + y_2(n) = 10/9\; 5^n u(n) - \frac{1}{9}(0.5)^n u(n) + 0.5^n u(n)$

$= 10/9\; 5^n u(n) + 8/9\; 0.5^n u(n)$ //

Zero input and zero state response:

The response due to the initial conditions alone (in the absence of i/p, $x(n)=0$) is called zero i/p response. The response due to the i/p alone (assuming that all the initial conditions are zero) is called zero state response.

Total response = zero i/p response + zero state response.

1. $y(n)+5y(n-1)+6y(n-2) = x(n-1)+2x(n)$, where $x(n)=u(n)$. The initial conditions are $y(-1)=1, y(-2)=0$

Find a) zero i/p response
b) zero state response
c) Total response.

$y(n)+5y(n-1)+6y(n-2) = x(n-1)+2x(n)$.

Taking z transform.

$$Y(z)+5[z^{-1}Y(z)+y(-1)]+6[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)] = [z^{-1}X(z)+x(-1)]+2X(z).$$

a) zero i/p response:

$$Y(z)+5[z^{-1}Y(z)+\underbrace{y(-1)}_{=1}]+6[z^{-2}Y(z)+z^{-1}\underbrace{y(-1)}_{=1}+\underbrace{y(-2)}_{=0}]=0$$

$$Y(z)+5z^{-1}Y(z)+5+6z^{-2}Y(z)+6z^{-1}=0.$$

$$Y(z)[1+5z^{-1}+6z^{-2}]+5+6z^{-1}=0$$

$$Y(z)[1+5z^{-1}+6z^{-2}] = -5-6z^{-1}$$

$$Y(z)=\frac{-5-z^{-1}}{1+5z^{-1}+6z^{-2}} = \frac{z(-5z-6)}{z^2+5z+6} = \frac{z(-5z-6)}{(z+2)(z+3)}$$

$$\frac{Y(z)}{z} = \frac{-5z-6}{(z+2)(z+3)} = \frac{A}{z+2}+\frac{B}{z+3}.$$

$A=4$, $B=-9$.

$$\therefore \frac{Y(z)}{z} = 4\frac{1}{z+2}-9\frac{1}{z+3}$$

$$Y(z) = 4\frac{z}{z+2}-9\frac{z}{z+3}$$

Taking inverse

$$y(n) = 4(-2)^n u(n) - 9(-3)^n u(n)$$

b) zero state response:

$$Y(z)+5\left[z^{-1}Y(z)+0\right]+6\left[z^{-2}Y(z)+0+0\right]= z^{-1}X(z)+2X(z).$$

$$Y(z)+5z^{-1}Y(z)+6z^{-2}Y(z) = z^{-1}X(z)+2X(z)$$

$$Y(z)\left[1+5z^{-1}+6z^{-2}\right] = \cancel{X(z)}\left[z^{-1}+2\right]X(z)$$

$$Y(z) = \frac{\left[z^{-1}+2\right]}{\left[1+5z^{-1}+6z^{-2}\right]}\cdot\frac{z}{z-1}$$

$$= \frac{z(1+2z)}{(z+2)(z+3)}\cdot\frac{z}{(z-1)}$$

$$\frac{Y(z)}{z} = \frac{z(1+2z)}{(z+2)(z+3)(z-1)} = \frac{A}{z+2}+\frac{B}{z+3}+\frac{C}{z-1}$$

$$z(1+2z) = A(z+3)(z-1)+B(z+2)(z-1)+C(z+2)(z+3)$$

$$A=-2 \quad , \quad B=\frac{15}{4} \quad , \quad C=1/4$$

$$\therefore \frac{Y(z)}{z} = -2\frac{1}{z+2}+\frac{15}{4}\frac{1}{z+3}+\frac{1}{4}\frac{1}{z-1}$$

$$\frac{15}{4}-9 \qquad \frac{15-36}{4}\cdot\frac{21}{4}$$

$$Y(z) = -2\frac{z}{z+2}+\frac{15}{4}\frac{z}{z+3}+\frac{1}{4}\frac{z}{z-1}$$

Taking inv. z-transform

$$y(n) = -2(-2)^n u(n)+\frac{15}{4}(-3)^n u(n)+\frac{1}{4}u(n)$$

Total response = zero i/p response + zero state response

$$= 4(-2)^n u(n) - 9(-3)^n u(n) - 2(-2)^n u(n) + \frac{15}{4}(-3)^n u(n) + \frac{1}{4}u(n)$$

$$= 2(-2)^n u(n) - \frac{21}{4}(-3)^n u(n) + \frac{1}{4}u(n)$$

Unit III

Fourier Representation for periodic discrete time Signal.

→ Discrete time Fourier Series.

Definition of DTFS:

DTFS representation of a periodic sequence $x(n)$ is given by

$$x(n) = \sum_{K=\langle N \rangle} X(K)\, e^{jK\Omega n} \longrightarrow ① \text{ (synthesis eqn)}$$

The inverse DTFS is

$$X(K) = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jK\Omega n} \longrightarrow ② \text{ (Analysis eqn.)}$$

$N \rightarrow$ Fundamental period

~~$\Omega \rightarrow$ Fundamental frequency~~

$K, n \in -(N/2 - 1)$ to $N/2 \longrightarrow$ If N is even.

$K, n \in -\frac{(N-1)}{2}$ to $\frac{(N-1)}{2} \longrightarrow$ If N is odd.

In the above eqns. ① and ②, $X(K)$ are known as DTFS coefficients. Eqn. ① is known as Synthesis eqn and ② is known as Analysis eqn.

Steps to find Fourier coefficients $X(K)$:

from the Function:

1) Find the value of Ω and N
2) Express the given function in exponential form.
3) Express the above fun. in terms of Ω, put it as eqn ①.

~~8) Express the above~~

4) Write the Synthesis eqn.

$$x(n) = \sum_{k=\langle N\rangle} X(k)\, e^{jk\Omega n}.$$

5) Select the range of k

$k \in -(N/2 - 1)$ to $N/2 \longrightarrow$ If N is even.
$k \in -\frac{(N-1)}{2}$ to $\frac{(N-1)}{2} \longrightarrow$ If N is odd

6) Expand the synthesis eqn. by putting the values of k. Put it as eqn ②.

7) Comparing ① and ②, we get the fourier coefficients X(k)

From the figure:

1) Find N, $\Omega = 2\pi/N$.
2) Select the range of n
3) Write the analysis eqn. $X(k) = \frac{1}{N} \sum_{n=\langle N\rangle} x(n)\, e^{-jk}$
4) Expand the above eqn. by putting the values of n

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Qn. Determine the DTFS coefficients of $x(n) = 3\cos(\pi/8\, n)$

Step (1): Find the value Ω and N

$$\Omega = \pi/8$$

$$N = \frac{2\pi}{\Omega} m = \frac{2\pi}{\pi/8} m = 16 \quad (m=1)$$

Step (2): Express the given fun. in terms of exponential

$$x(n) = 3 \cdot \frac{1}{2}\left[e^{j\pi/8 n} + e^{-j\pi/8 n}\right]$$

$$= \frac{3}{2} e^{j\pi/8 n} + \frac{3}{2} e^{-j\pi/8 n}$$

Step (3): Express the above fun. in terms of Ω

$$x(n) = \frac{3}{2} e^{j\Omega n} + \frac{3}{2} e^{-j\Omega n} \longrightarrow (1)$$

Step (4): Write the Synthesis eqn

$$x(n) = \sum_{K=\langle N \rangle} X(K)\, e^{jK\Omega n}$$

Step (5): Select the range of k

$K \in -(N/2 - 1)$ to $N/2$ (If N is even).

$K \in -(16/2 - 1)$ to $16/2$.

$K \in -7$ to 8.

Step (6): Expand the Synthesis eqn, by putting the values of K.

$$x(n) = \sum_{K=-7}^{8} X(K)\, e^{jK\Omega n}$$

$$= X(-7)\, e^{-j7\Omega n} + \ldots\ldots X(-1)\, e^{-j\Omega n} + X(0)\, e^{j0\Omega n} + X(1)\, e^{j\Omega n} + X(2)\, e^{j2\Omega n} + \ldots X(8)\, e^{j8\Omega n}$$

$$x(n) = X(-7)\, e^{-j7\Omega n} + \ldots X(-1)\, e^{-j\Omega n} + X(0) + X(1)\, e^{j\Omega n} + \ldots + X(8)\, e^{j8\Omega n} \longrightarrow ②$$

Comparing ① and ②.

$X(-1) = 3/2$

$X(1) = 3/2$.

$$\therefore X(K) = \begin{cases} 3/2, & K = \pm 1 \\ 0 & -7 \le K \le 8 \text{ and } K \ne \pm 1 \end{cases}$$

Qn. Determine the DTFS coefficients of $x(n) = \cos \pi/4\, n$.

1) $\Omega = \pi/4$

$N = \frac{2\pi}{\Omega} = \frac{2\pi}{\pi/4} = 8$ (even).

2) $x(n) = \frac{e^{j\pi/4 n}}{2} + \frac{e^{-j\pi/4 n}}{2}$.

3) $x(n) = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n}$. $\longrightarrow ①$

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4. $x(n) = \sum_{k=\langle N\rangle} X(k)\, e^{jk\Omega n}$

5. $k \in -(\frac{N}{2}-1)$ to $N/2$

$\in -1$ to 2

6. $x(n) = \sum_{k=-1}^{2} X(k)\, e^{jk\Omega n}$

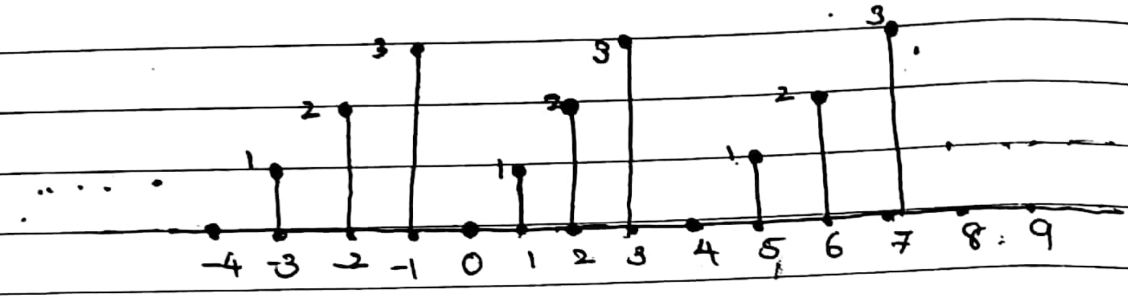
$= X(-1)\, e^{-j\Omega n} + X(0) + X(1)\, e^{j\Omega n} + X(2)\, e^{j2\Omega n} \longrightarrow ②$

7. Comparing ① and ②

$X(-1) = 1/2$
$X(1) = 1/2$

$\therefore X(k) = \begin{cases} 1/2 & k = \pm 1 \\ 0 & -1 \le k \le 2 \text{ and } k \ne \pm 1 \end{cases}$

Qn. Determine DTFS coefficients for the periodic s



1) Find N and Ω.

$N = 4$

$$\Omega = \frac{2\pi}{N} = \frac{2\pi}{4} = \pi/2.$$

2)

from the fig. $n \in 0$ to 3.

3) Write the analysis eqn.

$$X(K) = \frac{1}{N} \sum_{n \in \langle N \rangle} x(n) e^{-jK\Omega n}.$$

$$= \frac{1}{4} \sum_{n=0}^{3} x(n) e^{-jK\Omega n}.$$

$$= \frac{1}{4} \left[x(0) + x(1) e^{-jK\Omega} + x(2) e^{-j2K\Omega} + x(3) e^{-j3K\Omega} \right]$$

$$= \frac{1}{4} \left[0 + 1 e^{-jK\pi/2} + 2 e^{-j2K\pi/2} + 3 e^{-j3K\pi/2} \right]$$

$$x(k) = \frac{1}{4}\left[e^{-jk\pi/2} + 2e^{-jk\pi} + 3e^{-jk3\pi/2}\right]$$

$$x(0) = \frac{1}{4}\left[1+2+3\right] = \frac{6}{4} = 3/2 = 1.5$$

$$x(1) = \frac{1}{4}\left[e^{-j\pi/2} + 2e^{-j\pi} + 3e^{-j3\pi/2}\right]$$

$$= \frac{1}{4}\left[\cos\pi/2 - j\sin\pi/2 + 2(\cos\pi - j\sin\pi) + 3(\cos 3\pi/2 - j\sin 3\pi/2)\right]$$

$$= \frac{1}{4}\left[0 - j + 2(-1 - j\times 0) + 3(0 + j)\right]$$

$$= \frac{1}{4}\left[-j - 2 + 3j\right] = \frac{1}{4}\left[-2 + 2j\right] = -\frac{1}{2} + j\frac{1}{2}$$

$$= -0.5 + j0.5$$

$$x(2) = -1/2.$$

$$x(3) = -1/2 - j1/2 = -0.5 - 0.5j$$

$$\therefore x(k) = \{1.5, -0.5+j0.5, -0.5, -0.5\ j0.5\}.$$

Determine the DTFS representation of the sequence $x(n) = \cos^2(\pi/4\, n)$. Also sketch the magnitude spectrum.

$$x(n) = \cos^2(\pi/4\, n).$$

$$= \frac{1 + \cos 2\pi/4\, n}{2}.$$

$$= \frac{1 + \cos \pi/2\, n}{2}.$$

1) $\Omega = \pi/2$.

$$N = \frac{2\pi}{\Omega} m = \frac{2\pi}{\pi/2} m = 4 \quad (m=1).$$

2) $$x(n) = \frac{1}{2} + \frac{e^{j\pi/2 n}}{4} + \frac{e^{-j\pi/2 n}}{4}.$$

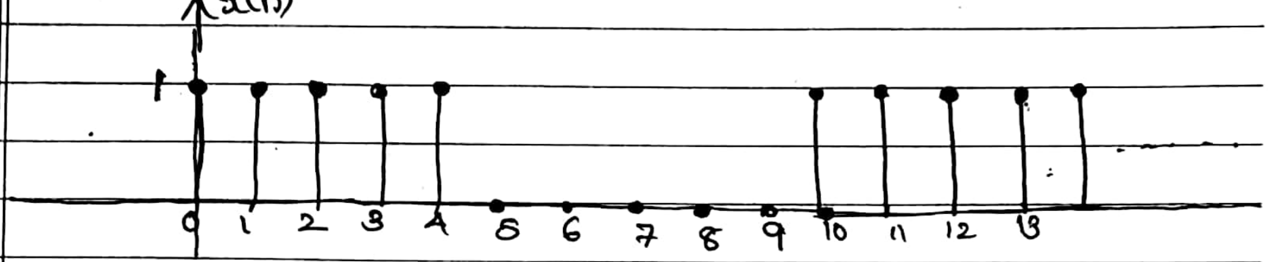
3) $$x(n) = \frac{1}{2} + \frac{e^{j\Omega n}}{4} + \frac{e^{-j\Omega n}}{4} \longrightarrow (1).$$

4) $$x(n) = \sum_{k=\langle N \rangle} X(k)\, e^{jk\Omega n}.$$

5) $k \in -\left(\frac{N}{2} - 1\right)$ to $\frac{N}{2}$.

$k \in -1$ to 2.

Determine the DTFS for $x(n)$. Also draw the magnitude spectrum.



1) From the figure $N = 10$

$$\Omega = \frac{2\pi}{N} = \frac{2\pi}{10} = \pi/5$$

2) n ranges from 0 to 9.

3) $$X(K) = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jK\Omega n}$$

$$= \frac{1}{10} \sum_{n=0}^{9} x(n)\, e^{-jK\Omega n}$$

$$= \frac{1}{10} \left\{ \sum_{n=0}^{4} x(n)\, e^{-jK\Omega n} + \sum_{n=5}^{9} x(n)\, e^{-jK\Omega n} \right\}$$

$$= \frac{1}{10} \left\{ \sum_{n=0}^{4} x(n)\, e^{-jK\Omega n} + 0 \right\}$$

5) $x(n) = \sum_{k=-1}^{2} X(k) e^{jk\Omega n}$

$= X(-1) e^{-j\Omega n} + X(0) + X(1) e^{j\Omega n} + X(2) e^{2j\Omega n} \longrightarrow (2)$

6) Comparing (1) and (2)

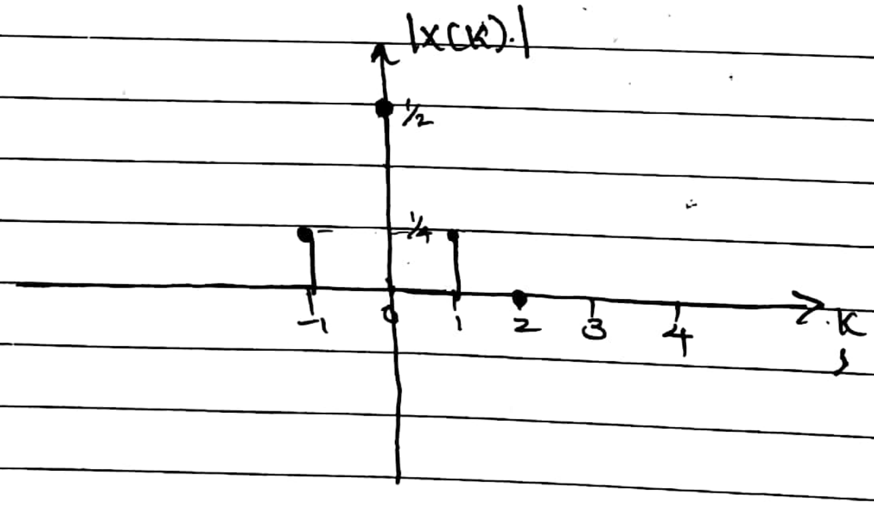
we get $X(-1) = \frac{1}{4}$

$X(0) = \frac{1}{2}$

$X(1) = \frac{1}{4}$

$X(2) = 0$

magnitude spectrum:



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$$= \frac{1}{10}\left[x(0) + x(1)e^{-j\Omega k} + x(2)e^{-j2\Omega k} + x(3)e^{-j3\Omega k} + x(4)e^{-j4\Omega k}\right]$$

$$X(k) = \frac{1}{10}\left[1 + x(1)e^{-j\Omega k} + x(2)e^{-j2\Omega k} + x(3)e^{-j3\Omega k} + x(4)e^{-j4\Omega k}\right]$$

$X(0) = 0.5$
$X(1) = 0.1 - j\,0.306$
$X(2) = 0$
$X(3) = 0.1 - j0.07$
$X(4) = 0$
$X(5) = 0.1$
$X(6) = 0$
$X(7) = 0.1 - j0.07$
$X(8) = 0$
$X(9) = 0.1 - j\,0.307$

Qn. Given DTFS coefficients $X(K) = \{0, 2, 0, -\tfrac{1}{2}, 0, 2, 0\}$ and $\Omega_0 = \frac{2\pi}{7}$. Find $x(n)$.

$$x(n) = \sum_{K=\langle N\rangle} X(K)\, e^{jK\Omega n}$$

$$= \sum_{K=-3}^{3} X(K)\, e^{jK\Omega n}$$

$$= X(-3)e^{-j3\Omega n} + X(-2)e^{-j2\Omega n} + X(-1)e^{-j\Omega n} + X(0) +$$

$$X(1)e^{j\Omega n} + X(2)e^{j2\Omega n} + X(3)e^{j3\Omega n}$$

$$= 2e^{-j2\Omega n} - \frac{1}{2} + 2e^{j2\Omega n}$$

$$= 2\left(e^{j2\Omega n} + e^{-j2\Omega n}\right) - \frac{1}{2}$$

$$= 4\cos(\Omega 2n) - \frac{1}{2}$$

$$= 4\cos\left(\frac{2\pi}{7}\cdot 2n\right) - \frac{1}{2}$$

$$= 4\cos\left(\frac{4\pi n}{7}\right) - \frac{1}{2}$$

H-W Sketch the magnitude and phase spectrum of $x(n) = 2\sin\frac{6\pi n}{7}$.

Properties of DTFS:

1) linearity:

If $x(n) \xleftrightarrow{DTFS} X(K)$

$y(n) \xleftrightarrow{DTFS} Y(K)$

then $z(n) = ax(n) + by(n) \xleftrightarrow{DTFS} Z(K) = aX(K) + bY(K)$

Proof: $Z(K) = \frac{1}{N} \sum_{n=\langle N \rangle} z(n)\, e^{-jK\Omega n}$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} [ax(n) + by(n)]\, e^{-jK\Omega n}$$

$$= \frac{1}{N} \left[\sum_{n=\langle N \rangle} a\, x(n)\, e^{-jK\Omega n} + \sum_{n=\langle N \rangle} b\, y(n)\, e^{-jK\Omega n} \right]$$

$$= a \underbrace{\frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega n}}_{X(K)} + b \underbrace{\frac{1}{N} \sum_{n=\langle N \rangle} y(n)\, e^{-jk\Omega n}}_{Y(K)}$$

$$= a X(K) + bY(K).$$

2. Time shift

If $x(n) \xleftrightarrow{DTFS} X(k)$.

then $z(n) = x(n-n_0) \xleftrightarrow{DTFS} Z(k) = e^{-jk\Omega n_0} X(k)$

Proof:

$$Z(k) = \frac{1}{N} \sum_{n=\langle N \rangle} z(n) e^{-jk\Omega n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n-n_0) e^{-jk\Omega n}$$

Put $m = n - n_0 \Rightarrow n = m + n_0$.

$$\therefore Z(k) = \frac{1}{N} \sum_{m=\langle N \rangle} x(m) e^{-jk\Omega(m+n_0)}.$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} x(m) e^{-jk\Omega m} \cdot e^{-jk\Omega n_0}$$

$$= e^{-jk\Omega n_0} \underbrace{\frac{1}{N} \sum_{m=\langle N \rangle} x(m) e^{-jk\Omega m}}_{X(k)}$$

$$= e^{-jk\Omega n_0} X(k).$$

3. Frequency shift:

If $x(n) \overset{DTFS}{\longleftrightarrow} X(K)$

then $z(n) = e^{jK_0\Omega n} x(n) \overset{DTFS}{\longleftrightarrow} Z(K) = X(K-K_0)$

Proof: $Z(K) = \frac{1}{N} \sum_{n=\langle N \rangle} z(n) e^{-jK\Omega n}$.

$= \frac{1}{N} \sum_{n=\langle N \rangle} e^{jK_0\Omega n} x(n) e^{-jK\Omega n}$.

$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(K-K_0)\Omega n}$.

$= X(K-K_0)$.

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5. Convolution:

If $x(n) \xleftrightarrow{DTFS} X(K)$

$y(n) \xleftrightarrow{DTFS} Y(K)$

then

$$z(n) = x(n) * y(n) \xleftrightarrow{DTFS} Z(K) = N\, X(K)\, Y(K)$$

Proof

$$Z(K) = \frac{1}{N} \sum_{n=\langle N \rangle} z(n)\, e^{-jK\Omega n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} [x(n) * y(n)]\, e^{-jK\Omega n}.$$

* We know that

$$x(n) * y(n) = \sum_{l=\langle N \rangle} x(l)\, y(n-l).$$

$$\therefore Z(K) = \frac{1}{N} \sum_{n=\langle N \rangle} \left[\sum_{l=\langle N \rangle} x(l)\, y(n-l) \right] e^{-jK\Omega n}$$

changing the order of summations

$$Z(K) = \frac{1}{N} \sum_{l=\langle N \rangle} x(l) \sum_{n=\langle N \rangle} y(n-l)\, e^{-jK\Omega n}.$$

Put $n-l = m \Rightarrow n = m+l$

$$\therefore Z(K) = \frac{1}{N} \sum_{l=\langle N \rangle} x(l) \sum_{m=\langle N \rangle} y(m)\, e^{-jK\Omega(m+l)}$$

$$Z(K) = \frac{1}{N} \sum_{l=\langle N\rangle} x(l) \sum_{m=\langle N\rangle} y(m)\, e^{-jK\Omega m}\, e^{-jK\Omega l}$$

$$= \underbrace{\frac{1}{N} \sum_{l=\langle N\rangle} x(l)\, e^{-jK\Omega l}}_{X(K)} \cdot \sum_{m=\langle N\rangle} y(m)\, e^{-jK\Omega m}.$$

$$= X(K) \cdot \frac{N}{N} \underbrace{\frac{1}{N} \sum_{m=\langle N\rangle} y(m)\, e^{-jK\Omega m}}_{Y(K)}.$$

$$= N\, X(K)\, Y(K).$$

6. Time reversal:

If $x(n) \xleftrightarrow{DTFS} X(K)$.

then $z(n) = x(-n) \xleftrightarrow{DTFS} Z(K) = X(-K)$

Proof: $$Z(K) = \frac{1}{N} \sum_{n=\langle N\rangle} z(n)\, e^{-jK\Omega n}$$

$$Z(K) = \frac{1}{N} \sum_{n=\langle N\rangle} x(-n)\, e^{-jK\Omega n}.$$

Put $m = -n$

$$\therefore Z(K) = \frac{1}{N} \sum_{m=\langle N\rangle} x(m)\, e^{+jK\Omega m}$$

$$= \frac{1}{N} \sum_{m=\langle N\rangle} x(m)\, e^{-j(-K)\Omega m}$$

$$= X(-K)$$

7. Symmetry:

If $x(n)$ is real

then $X(-K) = X(N-K) = X^*(K)$.

$x(n)$ can be written as sum of even & odd component,

$x(n) = x_e(n) + x_o(n)$

If $x(n) \xleftrightarrow{DTFS} X(K)$

then $x_e(n) \xleftrightarrow{DTFS} Re\{X(K)\}$

$x_o(n) \xleftrightarrow{DTFS} j\,I\{X(K)\}$.

ie if $x(n)$ is real and even, then its fourier coefficients are real. ~~and~~ while if $x(n)$ is real and odd, then its fourier coefficients are imaginary.

8. Conjugation.

If $x(n) \xleftrightarrow{DTFS} X(K)$

then ~~zoooo~~ $z(n) = x^*(n) \xleftrightarrow{DTFS}$ ~~zoooooo~~ $Z(K) = X^*(-K)$

Proof: ~~$Z(K) = \frac{1}{N} \sum_{n=\langle N\rangle} z(n) e^{-jK\Omega n}$~~.

~~$= \frac{1}{N} \sum_{n=\langle N\rangle} x^*(n) e^{-jK\Omega n}$~~.

$X(K) = \frac{1}{N} \sum_{n=\langle N\rangle} x(n) e^{jK\Omega n}$

$X^*(K) = \frac{1}{N} \left[\sum_{n=\langle N\rangle} x(n) e^{jK\Omega n}\right]^*$

$= \left[\frac{1}{N} \sum_{n=\langle N\rangle} x(n) e^{j(-K)\Omega n}\right]^*$

$= X^*(-K)$

$x(n) = \sum_{K=\langle N\rangle} X(K) e^{jK\Omega n}$.

$x^*(n) = \sum_{K=\langle N\rangle} X^*(K) e^{-jK\Omega n}$.

Put $l = -K$ $\therefore$ $x^*(n) = \sum_{l=\langle N\rangle} X^*(-l) e^{jl\Omega n}$.

$\therefore x^*(n) \xleftrightarrow{DTFS} X^*(-K)$.

9. Parseval's theorem:

If $x(n) \xleftrightarrow{DTFS} X(K)$

then average power, $P = \frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{K=\langle N \rangle} |X(K)|^2$

Proof:

$$P = \frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, x^*(n)$$

W.K.T $x(n) = \sum_{K=\langle N \rangle} X(K) e^{jK\Omega n}$

$$x^*(n) = \sum_{K=\langle N \rangle} X^*(K) e^{-jK\Omega n}.$$

$$\therefore P = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) \cdot \sum_{K=\langle N \rangle} X^*(K) e^{-jK\Omega n}.$$

changing the order of summations

$$P = \sum_{K=\langle N \rangle} X^*(K) \underbrace{\frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jK\Omega n}}_{X(K)}$$

$$P = \sum_{K=\langle N \rangle} X^*(K) X(K) = \sum_{K=\langle N \rangle} |X(K)|^2.$$

where $|X(K)|^2$ is the distribution of power as a function of frequency and is defined as the power spectral density spectrum of $x(n)$.

A plot of $|X(K)|^2$ Vs K is called power spectrum.

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Sketch the magnitude, phase and power spectrum of $x(n) = \cos\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)$

1) $\Omega' = \frac{6\pi}{13}$

$N = \frac{2\pi}{\Omega}m = \frac{2\pi}{3\,6\pi/13}m = \frac{13}{3}m = 13 \; (m=3)$

$\Omega = \frac{2\pi}{N} = \frac{2\pi}{13}$

2) $x(n) = \frac{e^{j\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)} + e^{-j\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)}}{2}$

$= \frac{e^{j\frac{6\pi}{13}n}}{2} \cdot e^{j\pi/6} + \frac{e^{-j\frac{6\pi}{13}n}}{2} \cdot e^{-j\pi/6}$

3) $x(n) = e^{j3\Omega n} \cdot \frac{e^{j\pi/6}}{2} + e^{-j3\Omega n} \frac{e^{-j\pi/6}}{2}$

$= \frac{e^{j\pi/6}}{2} e^{j3\Omega n} + \frac{e^{-j\pi/6}}{2} e^{-j3\Omega n} \longrightarrow (1)$

4) $x(n) = \sum_{k=\langle N \rangle} X(k) e^{jk\Omega n}$

5) $k \in -\frac{(N-1)}{2}$ to $\frac{(N-1)}{2}$

$\in -6$ to 6.

6) $x(n) = \sum_{k=-6}^{6} X(k) e^{jk\Omega n}$

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$$x(n) = X(-6)e^{-j6\Omega n} + \cdots X(-3)e^{-j3\Omega n} + X(-2)e^{-j2\Omega n}$$
$$+ X(-1)e^{-j\Omega n} + X(0) + X(1)e^{j\Omega n} + X(2)e^{j2\Omega n} + X(3)e^{j3\Omega n}$$
$$+ X(4)e^{j4\Omega n} + \cdots X(6)e^{j6\Omega n} \longrightarrow (2)$$

7) Comparing (1) and (2)

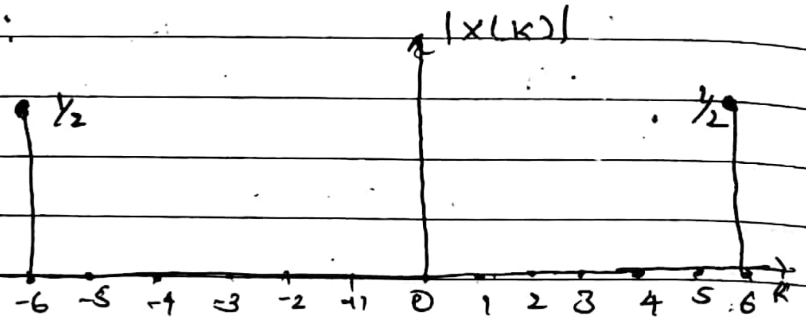
$$X(-3) = \frac{e^{-j\pi/6}}{2}$$

$$X(3) = \frac{e^{j\pi/6}}{2}.$$

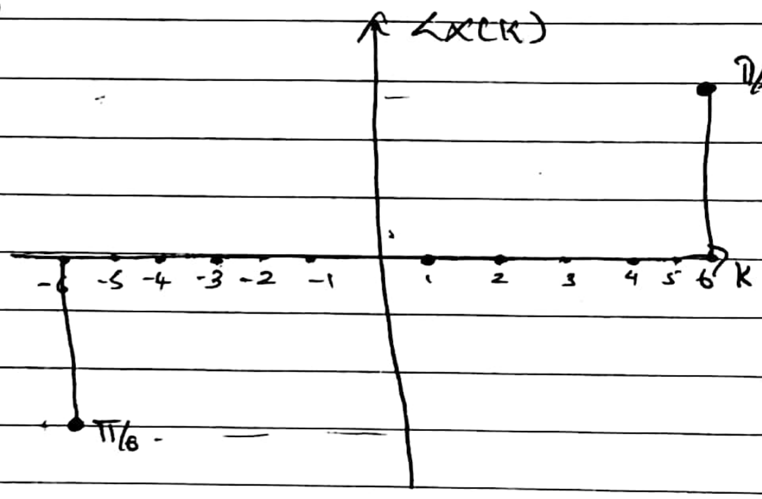
$$\therefore X(K) = \begin{cases} \frac{1}{2}e^{-j\pi/6} & K = -3. \\ \frac{1}{2}e^{j\pi/6} & K = 3 \\ 0 & \text{otherwise.} \end{cases}$$

K	X(K)	$\lvert X(K)\rvert$	$\angle X(K)$	$\lvert X(K)\rvert^2$
-6	0	0	0	0
-5	0	0	0	0
-4	0	0	0	0
-3	$\frac{1}{2}e^{-j\pi/6}$	$\frac{1}{2}$	$-\pi/6$	$\frac{1}{4}$
-2	0	0	0	0
-1	0	0	0	0
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	$\frac{1}{2}e^{j\pi/6}$	$\frac{1}{2}$	$\pi/6$	$\frac{1}{4}$
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0

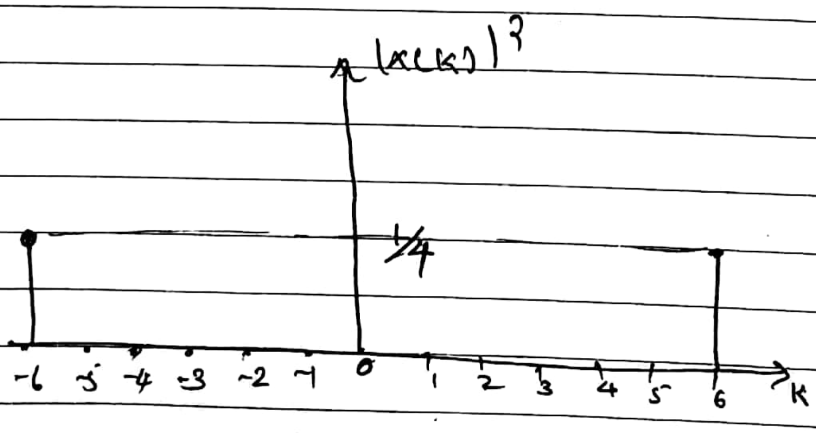
magnitude Spectrum :



phase spectrum



power Spectrum



Fourier representation for aperiodic discrete time Signal — Discrete time Fourier transform (DTFT).

Definition:

The DTFT of a non-periodic sequence $x(n)$ is given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} \longrightarrow \text{Analysis eqn.}$$

The inverse DTFT of $X(\Omega)$ is given by

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \longrightarrow \text{Synthesis eqn}$$

Amplitude and phase Spectrum:

A plot of $|X(\Omega)|$ Vs Ω is called amplitude Spectrum and a plot of $\angle X(\Omega)$ Vs Ω is called phase Spectrum.

Qn. Find the DTFT of unit impulse Sequence.

$$x(n) = \delta(n)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\Omega n}$$

$$= e^{-j\Omega n}\Big|_{n=0} = 1 \,//$$

Qn. Find the DTFT of $x(n) = \{\overset{x(0)}{1}, \overset{x(1)}{-1}, \overset{x(2)}{2}, \overset{x(3)}{2}\}$.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{3} x(n)\, e^{-j\Omega n}$$

$$= x(0) + x(1)\, e^{-j\Omega} + x(2)\, e^{-j2\Omega} + x(3)\, e^{-j3\Omega}$$

$$= \cancel{x(0)}\, 1 - 1e^{j\Omega} + 2e^{-j2\Omega} + 2e^{-j3\Omega}.$$

Qn. Find the inverse DTFT of $X(\Omega) = 1 + 2e^{-j\Omega} + 2e^{-j2\Omega} + 3e^{-j3\Omega}$ $\longrightarrow$ (1)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$X(\Omega) = \cdots\cdots x(-1)\, e^{j\Omega} + x(0) + x(1)\, e^{-j\Omega} + x(2)\, e^{-j2\Omega} + x(3)\, e^{-j3\Omega} + \cdots\cdots \longrightarrow (2)$$

Comparing (1) and (2)

$x(0) = 1$

$x(1) = 2$

$x(2) = 2$

$x(3) = 3$

$\therefore x(n) = \{1, 2, 2, 3\}.$

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Find the DTFT of the following

1) $x(n) = \{1, 2, 4, 6\}$

2) $x(n) = \{6, 7, 2, 1, 3\}$.

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Qn Find the DTFT of $x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$
Also find $X(\Omega)$ at $\Omega = 0$.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=-2}^{2} x(n) e^{-j\Omega n}$$

$$= x(-2) e^{+j2\Omega} + x(-1) e^{j\Omega} + x(0) + x(1) e^{-j\Omega} + x(2) e^{-j2\Omega}$$

$$= 1 \cdot e^{j2\Omega} + 2 e^{j\Omega} + 3 + 2 e^{-j\Omega} + 1 e^{-j2\Omega}$$

$$X(\Omega)\Big|_{\Omega=0} = 1+2+3+2+1 = 9$$

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Existence of DTFT

The DTFT does not exist for all aperiodic signals. A sufficient condition for the existence of DTFT for an aperiodic sequence $x(n)$ is

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

ie if a sequence $x(n)$ is absolutely summable, then DTFT exists for the sequence $x(n)$.

Qn. Find the DTFT of $x(n) = u(n)$.

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=-\infty}^{\infty} |u(n)| = \sum_{n=0}^{\infty} 1 = \infty$$

The seq. $x(n) = u(n)$ is not absolutely summable.

$\therefore$ DTFT does not exist.

Qn. Find the DTFT of $x(n) = a^n u(n)$.

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=-\infty}^{\infty} a^n u(n) = \sum_{n=0}^{\infty} a^n$$

$$= \frac{1}{1-a} < \infty \text{ for } |a| < 1$$

The sequence $x(n) = a^n u(n)$ is absolutely summable for all $|a| < 1$. $\therefore$ DTFT exists for $|a| < 1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (a e^{-j\Omega})^n = \frac{1}{1 - a e^{-j\Omega}}, \quad |a| < 1$$

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Properties of DTFT:

1) **linearity:**

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

$y(n) \xleftrightarrow{DTFT} Y(\Omega)$

then $z(n) = a x(n) + b y(n) \xleftrightarrow{DTFT} Z(\Omega) = a X(\Omega) + b Y(\Omega)$

Proof:

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [a x(n) + b y(n)] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} a x(n) e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} b y(n) e^{-j\Omega n}$$

$$= a \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} + b \sum_{n=-\infty}^{\infty} y(n) e^{-j\Omega n}$$

$$= a X(\Omega) + b Y(\Omega).$$

2) **Time shift:**

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then $z(n) = x(n-n_0) \xleftrightarrow{DTFT} Z(\Omega) = e^{-j\Omega n_0} X(\Omega)$

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} x(n-n_0) e^{-j\Omega n}$$

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Put $m = n - n_0 \Rightarrow n = m + n_0$.

$$\therefore Z(\Omega) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega(m+n_0)}$$

$$\sum_{m+n_0=-\infty}^{\infty}$$
$m = -\infty - n_0$
$= -\infty$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega m} \cdot e^{-j\Omega n_0}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega m}$$

$$= e^{-j\Omega n_0} X(\Omega).$$

3. Frequency shift:

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then $z(n) = e^{j\Omega_0 n} x(n) \xleftrightarrow{DTFT} X(\Omega - \Omega_0)$.

Proof: $$Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\Omega - \Omega_0)n}$$

$$= X(\Omega - \Omega_0).$$

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4. Scaling:

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

Then $z(n) = x(an) \xleftrightarrow{DTFT} Z(\Omega) = X(\Omega/a)$.

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} x(an) e^{-j\Omega n}.$$

Put $m = an \Rightarrow n = m/a$

$$Z(\Omega) = \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m/a}$$

$\sum_{m/a=-\infty}$
$m = a \times -\infty$
$= -\infty$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-j(\Omega/a)m}.$$

$$= X(\Omega/a).$$

5. Convolution:

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

$y(n) \xleftrightarrow{DTFT} Y(\Omega)$

then $z(n) = x(n) * y(n) \xleftrightarrow{DTFT} Z(\Omega) = X(\Omega).Y(\Omega)$

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} [x(n) * y(n)] e^{-j\Omega n}.$$

W.K.T $x(n) * y(n) = \sum_{l=-\infty}^{\infty} x(l)\, y(n-l)$

$$\therefore Z(\Omega) = \sum_{n=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} x(l)\, y(n-l) \right] e^{-j\Omega n}.$$

changing the order of summations

$$Z(\Omega) = \sum_{l=-\infty}^{\infty} x(l) \sum_{n=-\infty}^{\infty} y(n-l)\, e^{-j\Omega n}.$$

put $m = n-l \Rightarrow n = m+l.$

$$\therefore Z(\Omega) = \sum_{l=-\infty}^{\infty} x(l) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\Omega (m+l)}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\Omega m} \cdot e^{-j\Omega l}$$

$$= \underbrace{\sum_{l=-\infty}^{\infty} x(l)\, e^{-j\Omega l}}_{X(\Omega)} \cdot \underbrace{\sum_{m=-\infty}^{\infty} y(m)\, e^{-j\Omega m}}_{Y(\Omega).}$$

$$\therefore Z(\Omega) = X(\Omega) \cdot Y(\Omega).$$

6. Multiplication (modulation) :

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$
$y(n) \xleftrightarrow{DTFT} Y(\Omega)$

Then $z(n) = x(n) \cdot y(n) \xleftrightarrow{DTFT} Z(\Omega) = \frac{1}{2\pi} (X(\Omega) * Y(\Omega$

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}.$

$$= \sum_{n=-\infty}^{\infty} [x(n) \cdot y(n)]\, e^{-j\Omega n}.$$

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W.K.T $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\Omega_0) e^{j\Omega_0 n} d\Omega_0$

$\therefore Z(\Omega) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{2\pi} X(\Omega_0) e^{j\Omega_0 n} d\Omega_0\right] \cdot y(n) e^{-j\Omega n}$

changing the order of summation and integration,

$Z(\Omega) = \frac{1}{2\pi} \int_{2\pi} X(\Omega_0) \sum_{n=-\infty}^{\infty} y(n) e^{-j(\Omega-\Omega_0)n} d\Omega_0$

$= \frac{1}{2\pi} \int_{2\pi} X(\Omega_0) Y(\Omega - \Omega_0) d\Omega_0$

$= \frac{1}{2\pi} \left[X(\Omega) * Y(\Omega)\right]$

7. Conjugation

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

Then $z(n) = x^*(n) \xleftrightarrow{DTFT} Z(\Omega) = X^*(-\Omega)$

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$= \sum_{n=-\infty}^{\infty} x^*(n) e^{-j\Omega n} = \left[\sum_{n=-\infty}^{\infty} x(n) e^{j\Omega n}\right]^*$

$= \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j(-\Omega)n}\right]^*$

$= X^*(-\Omega)$

8. Time Reversal:

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then $z(n) = x(-n) \xleftrightarrow{DTFT} Z(\Omega) = X(-\Omega)$

proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} x(-n) e^{-j\Omega n}$$

put $m = -n$.

$$\therefore Z(\Omega) = \sum_{m=\infty}^{-\infty} x(m) e^{j\Omega m}.$$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-j(-\Omega)m}.$$

$$= X(-\Omega).$$

9. Symmetry:

If $x(n)$ is real, then $X(-\Omega) = X^*(\Omega)$.

$x(n)$ can be written as sum of even and odd components $x(n) = x_e(n) + x_o(n)$.

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$.

then $x_e(n) \xleftrightarrow{DTFT} Re\{X(\Omega)\}$

$x_o(n) \xleftrightarrow{DTFT} j\, I\{X(\Omega)\}$.

ie If $x(n)$ is real and even, then its DTFT is purely real and if $x(n)$ is real and odd then its DTFT is purely imaginary.

10. **Frequency differentiation** (multiplication by n)

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then $n\,x(n) \xleftrightarrow{DTFT} j\dfrac{dX(\Omega)}{d\Omega}$.

Proof: $X(\Omega) = \displaystyle\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$

differentiating both sides w.r.t. Ω.

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} \cdot (-jn).$$

$$\frac{1}{-j}\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} (n\,x(n))\, e^{-j\Omega n}$$

$$j\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\Omega n}$$

$\therefore\ n\,x(n) \xleftrightarrow{DTFT} j\dfrac{dX(\Omega)}{d\Omega}$.

$|x(n)|^2 = x(n) \cdot x^*(n)$

11. **Parseval's theorem:**

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then energy, $E = \displaystyle\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2\, d\Omega$

Proof:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot x^*(n).$$

W.K.T $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$

$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\Omega) e^{-j\Omega n} d\Omega$.

$\therefore E = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\Omega) e^{-j\Omega n} d\Omega$

changing the order of summation & integration

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\Omega) \underbrace{\sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}}_{X(\Omega)} d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\Omega) X(\Omega) d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega.$$

* In the above equ $|X(\Omega)|^2$ is known as energy ~~density spectra~~ spectral density of signal $x(n)$ and E is total energy content of sequence $x(n)$.

$$\boxed{1 \xleftrightarrow{DTFT} 2\pi\delta(\Omega).}$$

Qns. Find the DTFT of the following signals.

1. $x(n) = e^{j\Omega_0 n}$.

$x(n) = 1\, e^{j\Omega_0 n}$

By frequency shift property. $e^{j\Omega_0 n} \xleftrightarrow{DTFT} 2\pi\delta(\Omega - \Omega$

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2. $x(n) = \cos \Omega_0 n$.

$$= \frac{e^{j\Omega_0 n}}{2} + \frac{e^{-j\Omega_0 n}}{2}.$$

W.K.T $1 \xleftrightarrow{DTFT} 2\pi \delta(\Omega)$.

$$\left.\begin{aligned} 1.\frac{e^{j\Omega_0 n}}{2} &\xleftrightarrow{DTFT} \pi\delta(\Omega - \Omega_0) \\ 1.\frac{e^{-j\Omega_0 n}}{2} &\xleftrightarrow{DTFT} \pi\delta(\Omega + \Omega_0). \end{aligned}\right\} \text{By freq. shift property.}$$

$\therefore \cos \Omega_0 n \xleftrightarrow{DTFT} \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$.

3. $x(n) = n(1/2)^n u(n)$.

W.K.T $a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1 - ae^{-j\Omega}}$

$\therefore (1/2)^n u(n) \xleftrightarrow{DTFT} \frac{1}{1 - 1/2 e^{-j\Omega}}$

By multiplication by n property.

$$n\,x(n) \xleftrightarrow{DTFT} j\frac{d}{d\Omega} X(\Omega).$$

where $x(n) = (1/2)^n u(n)$, and $X(\Omega) = \frac{1}{1 - 1/2 e^{-j\Omega}}$

$\therefore n\,x(n) = n(1/2)^n u(n) \xleftrightarrow{DTFT} j\frac{d}{d\Omega} \frac{1}{1 - 1/2 e^{-j\Omega}}$

$$= j\frac{d}{d\Omega}\left[\frac{e^{+j\Omega}}{e^{j\Omega} - 0.5}\right]$$

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$$= \frac{j\left[(e^{j\Omega}-0.5)\cdot e^{j\Omega}(j) - e^{j\Omega}e^{j\Omega}(j)\right]}{(e^{j\Omega}-0.5)^2}$$

$$= \frac{-\cancel{e^{j2\Omega}} + 0.5e^{j\Omega} + \cancel{e^{j2\Omega}}}{(e^{j\Omega}-0.5)^2}$$

$$= \frac{0.5e^{j\Omega}}{(e^{j\Omega}-0.5)^2}$$

$$\boxed{n\,a^n u(n) \xleftrightarrow{DTFT} \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}}$$

Q $x(n) = (n+1)\,a^n u(n)$

$$= n a^n u(n) + a^n u(n) \xleftrightarrow{DTFT} \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2} + \frac{1}{1-ae^{-j\Omega}}$$

$$= \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2} + \frac{e^{j\Omega}}{(e^{j\Omega}-a)}$$

$$= \frac{ae^{j\Omega} + e^{j\Omega}(e^{j\Omega}-a)}{(e^{j\Omega}-a)^2} = \frac{\cancel{ae^{j\Omega}} + e^{j2\Omega} - \cancel{ae^{j\Omega}}}{(e^{j\Omega}-a)^2}$$

$$= \frac{e^{j2\Omega}}{(e^{j\Omega}-a)^2} = \frac{1}{(1-ae^{-j\Omega})^2}$$

$$\boxed{(n+1)\,a^n u(n) \xleftrightarrow{DTFT} \frac{1}{(1-ae^{-j\Omega})^2}}$$

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Find the DTFT of

5: $x(n) = (\frac{1}{2})^n u(n+1)$ by using property & using DTFT eqn.

Ans:

$x(n) = 2 \cdot (\frac{1}{2})^{n+1} u(n+1)$.

$x(n) \xleftrightarrow{DTFT} X(\Omega)$

$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\Omega_0} X(\Omega)$

$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\Omega}}$

$a^{n+1} u(n+1) \xleftrightarrow{DTFT} \frac{e^{j\Omega}}{1-ae^{-j\Omega}}$

$2 \cdot (\frac{1}{2})^{n+1} u(n+1) \xleftrightarrow{DTFT} \frac{2e^{j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$

using DTFT eqn:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} (\tfrac{1}{2})^n u(n+1) e^{-j\Omega n}$$

$$= \sum_{n=-1}^{\infty} (\tfrac{1}{2})^n e^{-j\Omega n}.$$

$$= (\tfrac{1}{2})^{-1} e^{j\Omega} + \sum_{n=0}^{\infty} (\tfrac{1}{2})^n e^{-j\Omega n}$$

$$= 2e^{j\Omega} + \sum_{n=0}^{\infty} (\tfrac{1}{2} e^{-j\Omega})^n.$$

$$= 2e^{j\Omega} + \frac{1}{1-\frac{1}{2}e^{-j\Omega}} = \frac{2e^{j\Omega}(1-\frac{1}{2}e^{-j\Omega})+1}{(1-\frac{1}{2}e^{-j\Omega})}$$

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$$= \frac{2e^{j\Omega} - \cancel{1} + \cancel{1}}{(1 - \frac{1}{2}e^{-j\Omega})}$$

$$= \frac{2e^{j\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}.$$

Qn. evaluate the following function without computing $X(\Omega)$, where $X(\Omega)$ is the DTFT of the sequence $x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 2\}$. a) $X(\Omega)$ at $\Omega = 0$

b) $\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$.

Ans: a) $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

$$X(\Omega)\Big|_{\Omega=0} = \sum_{n=-\infty}^{\infty} x(n) = \sum_{n=-2}^{2} x(n) = 1+2+3+2+2 = 10$$

b) $$\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega = 2\pi \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= 2\pi \sum_{n=-2}^{2} |x(n)|^2.$$

$$= 2\pi \left[1^2 + 2^2 + 3^2 + 2^2 + 2^2 \right]$$

$$= 2\pi \left[1+4+9+4+4 \right]$$

$$= 2\pi [22]$$

$$= 44\pi$$

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5: Qn. Find the convolution of the signals given below using Fourier transform. $x_1(n) = (\frac{1}{2})^n u(n)$, $x_2(n) = (\frac{1}{3})^n u(n)$

Ans: $y(n) = x_1(n) * x_2(n) \xleftrightarrow{DTFT} Y(\Omega) = X_1(\Omega) X_2(\Omega)$.
(convolution property)

W.K.T $a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\Omega}}$.

$\therefore x_1(n) = (\frac{1}{2})^n u(n) \xleftrightarrow{DTFT} X_1(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$

$x_2(n) = (\frac{1}{3})^n u(n) \xleftrightarrow{DTFT} X_2(\Omega) = \frac{1}{1-\frac{1}{3}e^{-j\Omega}}$.

$Y(\Omega) = X_1(\Omega) \cdot X_2(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} \cdot \frac{1}{1-\frac{1}{3}e^{-j\Omega}}$

$= \frac{A}{1-\frac{1}{2}e^{-j\Omega}} + \frac{B}{1-\frac{1}{3}e^{-j\Omega}}$

$1 = A(1-\frac{1}{3}e^{-j\Omega}) + B(1-\frac{1}{2}e^{-j\Omega})$

Put $e^{-j\Omega} = 3 \Rightarrow 1 = B(1-\frac{1}{2}\cdot 3)$.

$1 = B(1-3/2)$.

$1 = B(\frac{2-3}{2})$

$1 = -\frac{1}{2}B$

$B = -2$ //

$\Rightarrow 1 = A(1-\frac{1}{3}\cdot 2)$

$= A(1-2/3)$

$1 = A(\frac{3-2}{3})$.

$1 = \frac{1}{3}A$

$A = 3$ //

$= 2e^{j}$

$\therefore Y(\Omega) = 3\dfrac{1}{1-\frac{1}{2}e^{-j\Omega}} + -2\dfrac{1}{1-\frac{1}{3}e^{-j\Omega}}$.

Taking inverse DTFT on both sides,

$y(n) = 3.(\frac{1}{2})^n u(n) - 2.(\frac{1}{3})^n u(n)$.

Qn. Verify Parseval's theorem for the sequence $x(n) = (\frac{1}{2})^n u(n)$

Ans: Parseval's theorem: $\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$

LHS: $\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |(\frac{1}{2})^n u(n)|^2 = \sum_{n=0}^{\infty} (\frac{1}{2})^{2n}$

$= \sum_{n=0}^{\infty} (\frac{1}{4})^n = \dfrac{1}{1-\frac{1}{4}} = \dfrac{1}{\frac{4-1}{4}} = \dfrac{1}{3/4} = 4/3$

$= 1.33$

RHS: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$

W.K.T $a^n u(n) \overset{DTFT}{\longleftrightarrow} \dfrac{1}{1-ae^{-j\Omega}}$

$\therefore (\frac{1}{2})^n u(n) \overset{DTFT}{\longleftrightarrow} \dfrac{1}{1-\frac{1}{2}e^{-j\Omega}}$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\dfrac{1}{1-\frac{1}{2}e^{j\Omega}}\right|^2 d\Omega$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \dfrac{1}{(1-\frac{1}{2}e^{-j\Omega})} \times \dfrac{1}{(1-\frac{1}{2}e^{j\Omega})} d\Omega$.

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \dfrac{d\Omega}{(1-\frac{1}{2}e^{j\Omega} - \frac{1}{2}e^{-j\Omega} + \frac{1}{4})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \dfrac{1}{1+\frac{1}{4}-\frac{1}{2}(e^{j\Omega}+e^{-j\Omega}}d$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1.25 - \cos\Omega}\, d\Omega .$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1.25 - \left(\frac{1-\tan^2(\Omega/2)}{1+\tan^2(\Omega/2)}\right)}\, d\Omega .$$ $\boxed{\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}}$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\frac{1.25(1+\tan^2(\Omega/2)) - (1-\tan^2 \Omega/2)}{1+\tan^2(\Omega/2)}}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+\tan^2(\Omega/2)}{1.25 + 1.25\tan^2 \Omega/2 - 1 + \tan^2 \Omega/2}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+\tan^2 \Omega/2}{0.25 + 2.25\tan^2(\Omega/2)}\, d\Omega \longrightarrow (1)$$

Put $t = \tan(\Omega/2)$ $\qquad \Omega = -\pi \Rightarrow t = \tan(-\pi/2)$

diff. w. r to Ω. $\qquad = -\tan \pi/2$

$\Omega = \pi \Rightarrow t = \infty \qquad = -\infty$

$$\frac{dt}{d\Omega} = \sec^2 \Omega/2 \cdot \tfrac{1}{2} \Rightarrow d\Omega = \frac{2\, dt}{\sec^2 \Omega/2}$$

$$= \frac{2\, dt}{1+\tan^2 \Omega/2} = \frac{2\, dt}{1+t^2}.$$

$\frac{0.25}{2.25}$

$$(1) \Rightarrow = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{(1+t^2)}}{0.25+2.25t^2} \cdot \frac{\cancel{2}}{\cancel{(1+t^2)}}\, dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{2.25\left(t^2 + \frac{0.25}{2.25}\right)}\, dt$$

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$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{2.25\left(t^2 + \left(\frac{1}{3}\right)^2\right)} dt$$

$$= \frac{1}{2.25\pi} \int_{-\infty}^{\infty} \frac{1}{t^2 + (1/3)^2} dt$$

W.K.T

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{2.25\pi} \Big[3 \cdot \tan^{-1} 3t \Big]_{-\infty}^{\infty}$$

$$= \frac{1}{2.25\pi} \, 3 \left[\tan^{-1}(\infty) - \cancel{3} \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{2.25\pi} \, 3 \left[\pi/2 - (-\pi/2)\right]$$

$$= \frac{1}{2.25\cancel{\pi}} \cdot 3 \, \cancel{2}\cancel{\pi}/\cancel{2} = \frac{3}{2.25} = 1.33 = \text{LHS} //$$

Frequency response in discrete time system.

The frequency response of a linear time invariant discrete time system can be obtained by applying a spectrum of input sinusoids to the system.
The frequency response gives the magnitude response and phase response of the system to the spectrum of input sinusoids.

Let the impulse response of an LTI system be $h(n)$ and the input $x(n)$ to the system be $e^{j\Omega n}$. The output of the system $y(n)$ can be obtained by using convolution sum.

$$y(n) = h(n) * x(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$$

$$= \sum_{K=-\infty}^{\infty} h(K)\, e^{j\Omega(n-K)}$$

$$= \sum_{K=-\infty}^{\infty} h(K)\, e^{j\Omega n} \cdot e^{-j\Omega K}$$

$$= \underbrace{e^{j\Omega n}}_{i/p} \underbrace{\sum_{K=-\infty}^{\infty} h(K)\, e^{-j\Omega K}}_{H(\Omega)}$$

The quantity $H(\Omega)$ is the frequency response of the system.

$x(n) = e^{j\Omega n} \rightarrow \boxed{\text{LTI system } H(\Omega)} \rightarrow y(n) = x(n) \cdot H(\Omega)$

$$\boxed{H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}}$$

A plot of $|H(\Omega)|$ Vs Ω is called magnitude ~~spectrum~~ response and a plot of $\angle H(\Omega)$ Vs Ω is called phase response.

Qn. Find the frequency response of the system
$y(n) = \frac{1}{2}x(n) + x(n-1) + \frac{1}{2}x(n-2)$

Ans: Given $y(n) = \frac{1}{2}x(n) + x(n-1) + \frac{1}{2}x(n-2)$

Taking FT on both sides.

$$Y(\Omega) = \frac{1}{2}X(\Omega) + e^{-j\Omega}X(\Omega) + \frac{1}{2}e^{-j2\Omega}X(\Omega)$$
$$= X(\Omega)\left[\frac{1}{2} + e^{-j\Omega} + \frac{1}{2}e^{-j2\Omega}\right]$$

frequency response, $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{2} + e^{-j\Omega} + \frac{1}{2}e^{-j2\Omega}.$

$$= e^{-j\Omega}\left[\frac{1}{2}e^{j\Omega} + 1 + \frac{1}{2}e^{-j\Omega}\right]$$

$$= e^{-j\Omega}\left[1 + \cos\Omega\right]$$

Qn. Find the frequency response of the system
$y(n) - \frac{1}{4}y(n-1) - \frac{3}{8}y(n-2) = x(n) + x(n-1).$

Ans:

Taking DTFT on both sides.

$$Y(\Omega) - \frac{1}{4}e^{-j\Omega}Y(\Omega) - \frac{3}{8}e^{-j2\Omega}Y(\Omega) = X(\Omega) + e^{-j\Omega}X(\Omega)$$

$$Y(\Omega)\left[1 - \frac{1}{4}e^{-j\Omega} - \frac{3}{8}e^{-j2\Omega}\right] = X(\Omega)\left[1 + e^{-j\Omega}\right]$$

freq. response, $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 + e^{-j\Omega}}{\left[1 - \frac{1}{4}e^{-j\Omega} - \frac{3}{8}e^{-j2\Omega}\right]}$

Qn. Consider a causal and stable LTI s/m whose i/p $x(n)$ and o/p $y(n)$ are related through the 2nd order difference eqn: $y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$

a) Determine the frequency response $H(\Omega)$ of the s/m

b) Determine the impulse response of $h(n)$ of the s/m.

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Ans: Given $y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$

Taking FT on both sides.

$$Y(\Omega) - \frac{1}{6}e^{-j\Omega}Y(\Omega) - \frac{1}{6}e^{-j2\Omega}Y(\Omega) = X(\Omega)$$

$$Y(\Omega)\left[1 - \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}\right] = X(\Omega).$$

a) frequency response, $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1-\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}$

b) $H(\Omega) = \frac{1}{1-\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}} = \frac{1}{(1-\frac{1}{2}e^{-j\Omega})(1+\frac{1}{3}e^{-j\Omega})}$

$$= \frac{A}{(1-\frac{1}{2}e^{-j\Omega})} + \frac{B}{(1+\frac{1}{3}e^{-j\Omega})}$$

$$1 = A\left(1+\frac{1}{3}e^{-j\Omega}\right) + B\left(1-\frac{1}{2}e^{-j\Omega}\right)$$

Put $e^{-j\Omega} = -3 \Rightarrow 1 = B(1+3/2)$

$$\frac{5}{2}B = 1$$

$$B = 2/5$$

Put $e^{-j\Omega} = 2 \Rightarrow 1 = A(1+2/3)$

$$\frac{5}{3}A = 1$$

$$A = 3/5$$

$\therefore H(\Omega) = \frac{3}{5}\frac{1}{(1-\frac{1}{2}e^{-j\Omega})} + \frac{2}{5}\frac{1}{(1+\frac{1}{3}e^{-j\Omega})}$

Taking inverse DTFT

impulse response, $h(n) = \frac{3}{5}\left(\frac{1}{2}\right)^n u(n) + \frac{2}{5}\left(-\frac{1}{3}\right)^n u(n)$

$$\left(\because a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\Omega}}\right.$$

Qn. Consider a discrete time LTI s/m with impulse respon $h(n) = (\frac{1}{2})^n u(n)$. Use Fourier transform determine the response to the following signals.

a) $x(n) = (\frac{3}{4})^n u(n)$

b) $x(n) = (-1)^n u(n)$.

Ans:

Given $h(n) = (\frac{1}{2})^n u(n)$

$$H(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}.$$

a) $x(n) = (\frac{3}{4})^n u(n)$

$$X(\Omega) = \frac{1}{1 - \frac{3}{4}e^{-j\Omega}}$$

$$Y(\Omega) = H(\Omega) \cdot X(\Omega)$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\Omega}} \cdot \frac{1}{1-\frac{3}{4}e^{-j\Omega}} = \frac{A}{1-\frac{1}{2}e^{-j\Omega}} + \frac{B}{1-\frac{3}{4}e^{-j\Omega}}$$

$A = 3$, $B = -2$

$$\therefore Y(\Omega) = 3 \cdot \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - 2\frac{1}{1-\frac{3}{4}e^{-j\Omega}}.$$

Taking inverse DTFT

$$y(n) = 3 (\tfrac{1}{2})^n u(n) - 2 (\tfrac{3}{4})^n u(n) //$$

b) $x(n) = (-1)^n u(n)$

$$X(\Omega) = \frac{1}{1+e^{-j\Omega}}.$$

$$Y(\Omega) = H(\Omega) \cdot X(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} \cdot \frac{1}{1+e^{-j\Omega}} = \frac{A}{1-\frac{1}{2}e^{-j\Omega}} + \frac{B}{1+e^{-j\Omega}}$$

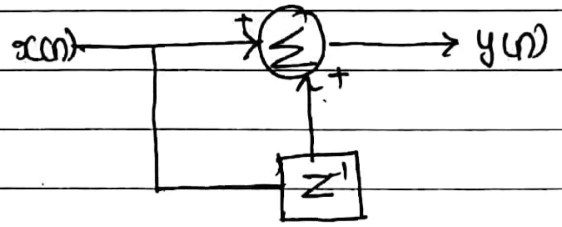
$A = 2/5$, $B = 1/3$

$$\therefore Y(\Omega) = 2/5 \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{3} \frac{1}{1+e^{-j\Omega}}.$$

Taking inverse DTFT

$$y(n) = 2/5 (\tfrac{1}{2})^n u(n) + \tfrac{1}{3} (-1)^n u(n) //$$

Qn. Consider the discrete time LTI S/m shown in fig. given below.



a) Find the frequency response of the s/m.
b) Find the impulse response of the s/m.
c) Find the 3dB Bandwidth of the s/m.
d) Sketch the magnitude and phase responses.

Ans: from the fig. $y(n) = x(n) + x(n-1)$

$$Y(\Omega) = X(\Omega) + e^{-j\Omega} X(\Omega)$$
$$= X(\Omega)[1 + e^{-j\Omega}]$$

a) frequency response, $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = 1 + e^{-j\Omega}$ //

$$= 1 + \cos\Omega - j\sin$$

b) Impulse response means when $x(n) = \delta(n)$, then the $y(n) = h(n)$.

$\therefore h(n) = \delta(n) + \delta(n-1)$.

c) 3 dB Bandwidth

$$|H(\Omega)_{3dB}| = \frac{1}{\sqrt{2}} |H(0)|$$

$$= \frac{1}{\sqrt{2}} 2 = \sqrt{2}.$$

$$\sqrt{(1+\cos\Omega_{3dB})^2 + \sin^2\Omega_{3dB}} = \sqrt{2}.$$

$\sqrt{1+2\cos\Omega_{3dB}+\underbrace{\cos^2\Omega_{3dB}+\sin^2\Omega_{3dB}}_{=1}} = \sqrt{2}$

$\sqrt{2+2\cos\Omega_{3dB}} = \sqrt{2}$

$2+2\cos\Omega_{3dB} = 2.$

$2\cos\Omega_{3dB} = 0.$

$\cos\Omega_{3dB} = 0 \Rightarrow \Omega_{3dB} = \cos^{-1}(0) = \pi/2$ //.

$\cos\pi = -1$
$\cos\pi/2 = 0$
$\cos 0 = 1$
$\cos\pi/4 = 1/\sqrt{2}$

d)

Ω	$\lvert H(\Omega)\rvert = \sqrt{2+2\cos\Omega}$	$\angle H(\Omega) = \tan^{-1}\left(\frac{-\sin\Omega}{1+\cos\Omega}\right) = -\tan^{-1}\left(\frac{\sin\Omega}{1+\cos\Omega}\right)$
$-\pi$	0	$\tan^{-1}(0/0) = \tan^{-1}(\infty) = 90$
$-\pi/4$	1.84	+22.5
$-\pi/2$	1.414	+45
0	2	0
$\pi/2$	1.414	-45
$\pi/4$	1.84	-22.5
π	0	-90